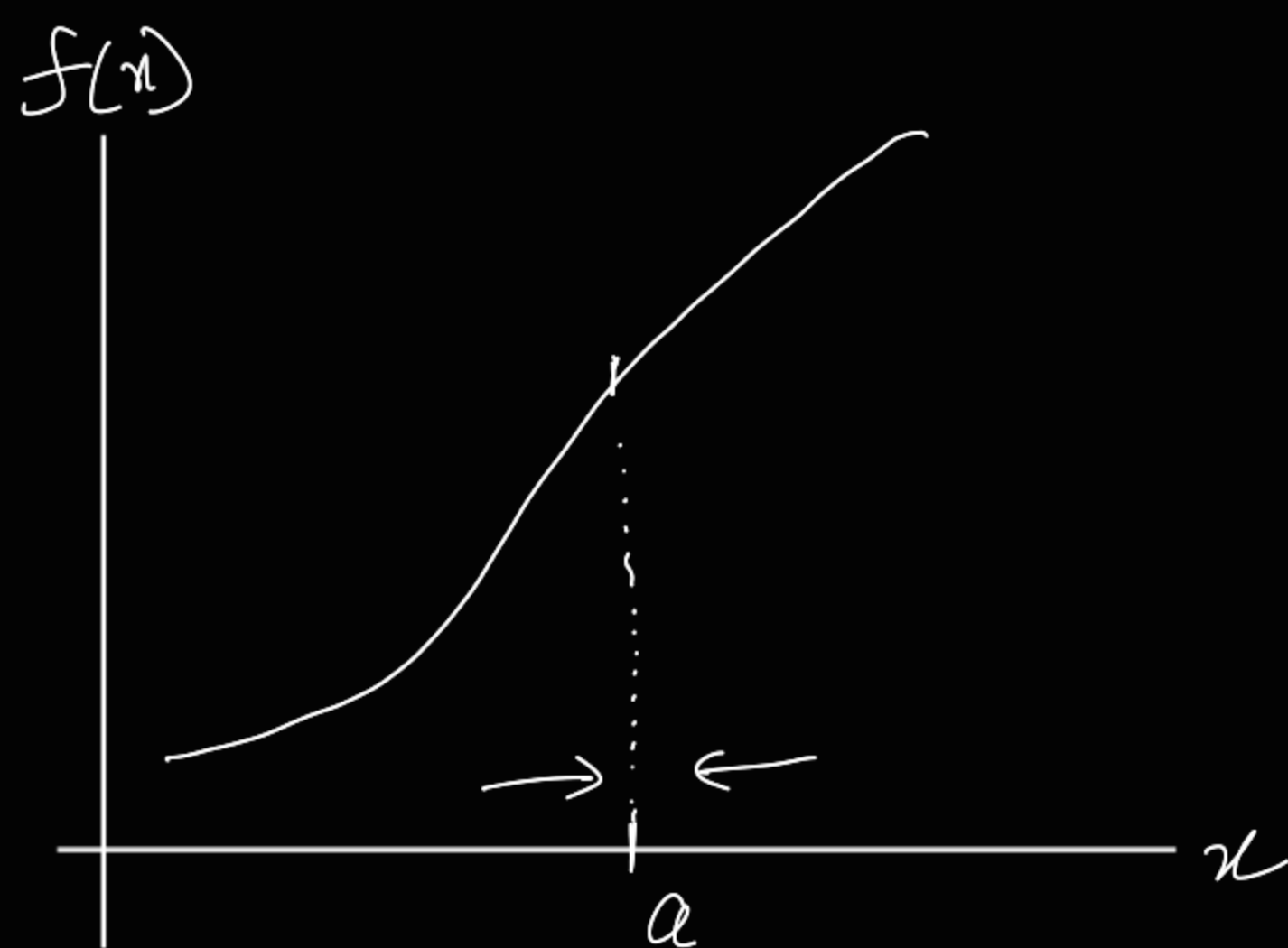


Calculus & Optimisation



$$\lim_{x \rightarrow a} f(x) \quad x \rightarrow a \neq x = a$$

approach from left
approach from right

eg:- $f(x) = \frac{(x-2)(x-3)}{(x-2)}$ at $x=2 = \frac{0}{0}$ form
undetermined

but $\lim_{x \rightarrow 2} \frac{(x-2)(x-3)}{(x-2)}$ } we can cancel inside limit only.

→ We are not interested in the exact value but only the left & right limit. ⇒

eg $\lim_{x \rightarrow 0} \frac{(x+1)^2 - 1}{x} = 2$

Value at $f(2) =$ not defined
but $\lim_{x \rightarrow 0} f(x) = 2$

x	$f(x)$
-0.01	1.99
-0.001	1.999
-0.0001	1.9999
0.0001	2.0001
0.0001	2.001
0.001	2.01

LHL → Left hand Limit
RHL → Right hand Limit

$$\lim_{x \rightarrow a} f(x)$$

LHL =
RHL =

$$\lim_{h \rightarrow 0} f(x-h)$$

also written as $\lim_{x \rightarrow a^-} f(x)$

approach from left of Numberline

$$\lim_{h \rightarrow 0} f(x+h)$$

approach from right of Numberline

$$\lim_{x \rightarrow a^+} f(x)$$

Limit $f(x)$ exist only if LHL & RHL exists & $LHL = RHL$

Limit doesn't exist

- LHL or RHL is ∞ or Not defined
- $LHL \neq RHL$
- periodic function at infinity

Limit might be different from the value of function at that point

$$\lim_{x \rightarrow a} f(x) \text{ need not be } f(a) \text{ even if } f(a) \text{ exists.}$$

need not be equal to
but can be

Indeterminate forms in Limit

- IM { 1) $0 \cdot \infty \rightarrow 0 \cdot \infty \rightarrow 0^+ \cdot \infty \text{ or } 0^- \cdot \infty$
- 2) $0/0 \rightarrow 0/0^+ \text{ or } 0/0^-$
- 3) $\frac{\infty}{\infty}$
- 4) $\infty - \infty$
- 5) 0^0
- 6) ∞^0
- 7) 1^∞
- any other form.

- EXAMPLES
- These are not indeterminate forms
- $\frac{0}{\infty} = 0$
 - $\infty \cdot \infty = \infty$
 - $-\infty \cdot \infty = -\infty$
 - $\frac{\infty}{0} = \infty$
 - $\infty + \infty = \infty$
 - $0^\infty \rightarrow \text{determinate}$
since $0^\infty \rightarrow 0$ No conflict.
 - $\infty^\infty \rightarrow \infty$

Product

$$\left. \begin{array}{l} \lim_{x \rightarrow a} f(x) \cdot g(x) \\ \text{if in determinate form} \end{array} \right\} \begin{array}{l} \text{if } f(x) \rightarrow 3^+ \\ \text{and } g(x) \rightarrow 5^+ \end{array}$$

$$= \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

provided they are not in the indeterminate form.

In Indeterminate forms, the Limit depends on the dominating factors.

Indeterminate vs undefined

We need more analysis like log, L'Hos.

Not defined $\Rightarrow$ Limit does not exist

Limit of ∞ is a very specific way of saying Limit does not exist.

Methods to evaluate Limit

- $\rightarrow$ Solving $0/0$
- Factorization
 - Rationalization
 - Completing the square
 - Simplifying using identities $\left. \begin{array}{l} a^2 - b^2 \\ (a+b)^2 \\ (a+b)^3 \end{array} \right\} \text{etc}$
 - Conjugate Rational multi
 - Polynomial division

- $\rightarrow$ Use of standard Limits
- $\rightarrow$ Solving ∞/∞
- $\rightarrow$ Solving $\infty - \infty$
- $\rightarrow$ Solving 1^∞
- $\rightarrow$ L'Hospital's rule

Factorization

$$Q) \lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = \frac{x^2(5x+8)}{x^2(3x^2-16)} \Rightarrow \lim_{x \rightarrow 0} f(x) = \frac{8}{-16} = \underline{\underline{-1/2}}$$

$$* Q) \lim_{x \rightarrow 0} \frac{(x+2)^3 - 8}{x} = \frac{(x+2)(x^2+4x+4) - 8}{x} = \frac{x^3 + 6x^2 + 12x + 8 - 8}{x} = \frac{x^3 + 6x^2 + 12x}{x} = \underline{\underline{12}}$$

$$Q) \lim_{x \rightarrow 4} \frac{2x^2 - 12x + 16}{x^2 - x - 12} = \frac{2(x-4)(x-2)}{(x-4)(x+3)} = \frac{2 \cdot (2)}{1} = \underline{\underline{4}}$$

$$Q) \lim_{x \rightarrow -3} \frac{x^2 + 2x - 3}{-3x - 9} = \frac{(x+3)(x-1)}{-3(x+3)} = \frac{-4}{-3} = \underline{\underline{4/3}}$$

$$* Q) \lim_{x \rightarrow 16} \frac{16-x}{4-\sqrt{x}} = \lim_{x \rightarrow 16} \frac{(16-x)(4+\sqrt{x})}{(16-x)} = 8 // \text{Rationalisation} \rightarrow \text{multiplying by conjugate rational}$$

$$Q) \lim_{x \rightarrow 25} \frac{5-\sqrt{x}}{25-x} = \lim_{x \rightarrow 25} \frac{25-x}{(25-x)(5+\sqrt{x})} = \frac{1}{5+5} = \underline{\underline{1/10}}$$

$$Q) \lim_{x \rightarrow 1} \frac{(x^3-1)}{(x-1)} = \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \underline{\underline{3}}$$

$$\begin{array}{r} x^2 + x + 1 \\ x^3 - 1 \\ \hline 0 \quad x^2 - 1 \\ \quad x^2 - x \\ \hline 0 \quad (x-1) \end{array}$$

Factorization of polynomial

$$Q) \lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1} = \frac{(\sqrt{x}-1)(x\sqrt{x} + x + \sqrt{x})}{(\sqrt{x}-1)}$$

$$\lim_{x \rightarrow 1} (x\sqrt{x} + x + \sqrt{x}) = \underline{\underline{3}}$$

$$\begin{array}{r} x\sqrt{x} + x + \sqrt{x} \\ x^2 - \sqrt{x} \\ \hline 0 \quad x\sqrt{x} - \sqrt{x} \\ \quad x\sqrt{x} - x \\ \hline 0 \quad x - \sqrt{x} \\ \quad x - \sqrt{x} \\ \hline 0 \end{array}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + b^2) + ab(a-b)$$

$$a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$$

$$Q) \lim_{x \rightarrow 1} \frac{(x+1)(\sqrt{x}-1)}{2x^2 + x - 3} = \frac{(x+1)(\sqrt{x}-1)}{2(x+\frac{3}{2})(x-1)} = \frac{(x+1)(x-1)}{2(x+\frac{3}{2})(x-1)(\sqrt{x}+1)} = \frac{2}{2(\frac{5}{2}) \cdot 2} = \underline{\underline{1/5}}$$

Conjugate rational multiplication

Rationalisation

$$Q \quad \lim_{x \rightarrow 0} \frac{\sqrt{x^2+1} - 1}{x} = \lim_{x \rightarrow 0} \frac{x^2+1-1}{x(\sqrt{x^2+1}+1)} = \frac{x^2}{x(\sqrt{x^2+1}+1)} = 0 //$$

$$Q \quad \lim_{t \rightarrow 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t \rightarrow 0} \frac{t^2}{t^2(\sqrt{t^2+9}+3)} = \frac{1}{6}$$

$$Q \quad \lim_{x \rightarrow -4} \frac{\sqrt{x^2+9} - 5}{x+4} = \lim_{x \rightarrow -4} \frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)} = \frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)} = \frac{-8}{10} = -\frac{08}{10}$$

$$Q \quad \lim_{x \rightarrow 0} \frac{\sqrt{2x+1} - \sqrt{3x+1}}{x} = \lim_{x \rightarrow 0} \frac{2x+1-(3x+1)}{x(\sqrt{2x+1}+\sqrt{3x+1})} = \frac{-1}{1+1} = -\frac{1}{2}$$

$$Q \quad \lim_{x \rightarrow 0} \frac{\sqrt{x^2+9} - \sqrt{2x^2+9}}{\sqrt{3x^2+4} - \sqrt{2x^2+4}} = \lim_{x \rightarrow 0} \frac{x^2+9-(2x^2+9) \cdot (\sqrt{3x^2+4} + \sqrt{2x^2+4})}{(3x^2+4)-(2x^2+4)(\sqrt{x^2+9} + \sqrt{2x^2+9})} = -\frac{(2+2)}{(3+3)} = -\frac{4}{6} = -\frac{2}{3}$$

$$Q \quad \lim_{x \rightarrow 0} \frac{x^3}{\sqrt{1+x^3} - \sqrt{1-x^3}} = \lim_{x \rightarrow 0} \frac{x^3(\sqrt{1+x^3} + \sqrt{1-x^3})}{1+x^3-(1-x^3)} = \frac{1}{2} \cdot (1+1) = 1 //$$

Properties of Limit

Let, $\lim_{x \rightarrow a} f(x) = l$ & $\lim_{x \rightarrow a} g(x) = m$

l, m should be finite not ∞

$$(1) \quad \lim_{x \rightarrow a} f(x) + g(x) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = l + m$$

$$(2) \quad \lim_{x \rightarrow a} f(x) \cdot g(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = l \times m$$

$$(3) \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{l}{m}; \boxed{m \neq 0}$$

$$(5) \quad \lim_{x \rightarrow a} f(x)^{g(x)} = \left[\lim_{x \rightarrow a} f(x) \right]^{\lim_{x \rightarrow a} g(x)} = l^m \quad \boxed{\text{if } l > 0}$$

$$(6) \quad \lim_{x \rightarrow a} f \circ g(x) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(m); \text{ provided } f(x) \text{ is continuous at } g(a) = m$$

STANDARD LIMIT

$$\bullet \star \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\bullet \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln(a)$$

$$\bullet \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \quad \bullet \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$\rightarrow \text{result} \quad \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$$

$$\rightarrow \lim_{x \rightarrow \infty} x \sin(1/x) = 1$$

other form

$$\rightarrow \lim_{x \rightarrow \infty} x \cdot \ln(1 + 1/x) = 1$$

$$\rightarrow \lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

$$\textcircled{Q} \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$$

$$\textcircled{Q} \lim_{x \rightarrow 0} \frac{\sin 8x}{x} = 8$$

$$\textcircled{Q} \lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} \cdot x = 0$$

$$\textcircled{Q} \star \star \star \lim_{x \rightarrow 0} \frac{15^x - 5^x - 3^x + 1}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{(5^x - 1)(3^x - 1)}{x^2} \cdot \frac{x^2}{\sin^2 x} = \ln(5) \cdot \ln(3)$$

$$e^a \cdot e^b = e^{a+b}$$

$$a = \ln x$$

$$b = \ln y$$

$$\ln(xy) = a + b$$

$$\ln(xy) = \ln x + \ln y$$

$$\textcircled{Q} \star \lim_{x \rightarrow 0} \frac{p(e^x - 2)}{2x} = 1 \quad \text{find } p \text{ and } q$$

$$\text{solution} \quad \lim_{x \rightarrow 0} \frac{pq(\frac{1}{2}e^x - 1)}{2x} = 1$$

$$\text{if } p = 2$$

$$2 = 1$$

$$\Rightarrow$$

$$pq = 3$$

$$\textcircled{Q} \lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$$

$$\textcircled{Q} \lim_{x \rightarrow 0} \frac{\sin 8x}{\sin 3x} = \frac{\sin 8x}{8x} \cdot \frac{8x}{3x} \cdot \frac{3x}{\sin 3x} = 8/3$$

$$\textcircled{Q} \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0 // \text{ not an indeterminate form}$$

Solving $\infty/\infty \rightarrow$ Take highest power common

$$Q) \lim_{x \rightarrow \infty} \frac{5x^3 - 2x^2 - 1}{x^3 - x + 1} = \lim_{x \rightarrow \infty} \frac{x^3(5 - \frac{2}{x} - \frac{1}{x^3})}{x^3(1 - \frac{1}{x^2} + \frac{1}{x^3})} = \frac{5}{1} = 5 //$$

$$Q) \lim_{x \rightarrow \infty} \frac{x-2}{\sqrt{2x^2-x+1}} = \frac{x(1-2/x)}{x\sqrt{2-\frac{1}{x}+\frac{1}{x^2}}} = \frac{1}{\sqrt{2}} //$$

$$Q) \lim_{x \rightarrow -\infty} \frac{x-2}{\sqrt{2x^2-x+1}} = \frac{x(1-2/x)}{|x|\sqrt{2-\frac{1}{x}+\frac{1}{x^2}}} = \frac{-1}{\sqrt{2}} \quad \left. \begin{array}{l} \text{While taking } x \text{ outside} \\ \text{the sq. rt when the} \\ \text{limit is } -\infty \text{ should} \\ \text{be very careful.} \end{array} \right\}$$

$$Q) \lim_{x \rightarrow \infty} \frac{x^2+1}{x^3-3} = \frac{x^2(1+1/x^2)}{x^3(1-3/x^3)} = 0$$

$$Q) \lim_{x \rightarrow 1} \frac{(x^3-1)}{(x^2-1)} = \frac{(x-1)(x^2+1+x)}{(x-1)(x+1)} = \frac{3}{2} //$$

$$Q) \lim_{x \rightarrow 0} \frac{x^2+2x+1}{3x^2-4x+5} = 1/5$$

$$Q) \lim_{n \rightarrow \infty} \frac{n^2}{1+2+3+\dots+n} = \frac{n^2 \cdot 2}{n(n+1)} = \frac{2n^2}{n^2(1+1/n)} = 2$$

$$Q) \lim_{x \rightarrow -\infty} \frac{\sqrt{3x^2+2}}{x-2} \quad \text{put } t = -x \Rightarrow \lim_{t \rightarrow \infty} \frac{t\sqrt{3+2/t^2}}{-t(1+2/t)} = \frac{-\sqrt{3}}{1}$$

$$Q) \lim_{x \rightarrow -\infty} \frac{2x^4-7x^2}{5x^6+8} = 0$$

$$Q) \lim_{x \rightarrow \infty} \frac{\sqrt{x^6-6x^3-3}}{10x^3+10} = \frac{x^3\sqrt{1-\frac{6}{x^3}-\frac{3}{x^6}}}{x^3(10+10/x^3)} = \frac{1}{10} //$$

$$Q) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^6-6x^3-3}}{10x^3+10}$$

Conv. $t = -x$

$$\lim_{t \rightarrow \infty} \frac{\sqrt{t^6+6t^3-3}}{-10t^3+10} = \frac{t^3\sqrt{1+\frac{6}{t^3}-\frac{3}{t^6}}}{t^3(-10+\frac{10}{t^3})} = \frac{-1}{10}$$

Solving $\infty - \infty$ form $\rightarrow$ Clubbing terms to reduce to $\frac{\infty}{\infty}$ form

Use rationalisation by conjugate multiplication

$$Q) \lim_{x \rightarrow \infty} (\sqrt{x^2+2x} - x) = \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x} = \frac{2x}{x\sqrt{1+2/x}+x} = \frac{2}{1+1} = 1 //$$

$$\textcircled{Q} \lim_{x \rightarrow \infty} \sqrt{x^2 - 2x + 1} - x = \lim_{x \rightarrow \infty} \frac{x^2 - 2x + 1 - x^2}{\sqrt{x^2 - 2x + 1} + x} = \frac{-2}{1+1} = \underline{\underline{-1}}$$

$$\textcircled{Q} \lim_{x \rightarrow \infty} \sqrt{x^2 - 2x + 1} - x = \lim_{t \rightarrow \infty} \frac{t(2 + \frac{1}{t})}{t\sqrt{1 - \frac{2}{t} + \frac{1}{t^2}}} - t = \frac{2}{1-1} = \frac{2}{0} = \infty$$

$$\textcircled{Q} \lim_{x \rightarrow \infty} \sqrt{x^2 + x} - \sqrt{x^2 - x} = \frac{(x^2 + x) - (x^2 - x)}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} = \frac{2x}{x(\sqrt{1 + 1/x} + \sqrt{1 - 1/x})} = \underline{\underline{1}}$$

$$\textcircled{Q} \star \lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 1}{x+1} - ax - b \right) = 4 \Rightarrow \frac{x^2 + x + 1}{x+1} - a(x+1)x = 4 + b$$

$$\Rightarrow \frac{x(1 + 1/x - a)}{x(1 + 1/x)} \Rightarrow a = \underline{\underline{1}} \rightarrow \text{critical ide}$$

$$\Rightarrow 1 - a = 4 + b \Rightarrow b = \underline{\underline{-4}}$$

L' Hopital rule

French

L' → The

H → silent

phonous → L'hopital

If the $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ results in one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\text{then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Note! $\star$
only apply if in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form in limits

$$\text{eg. 1) } \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{1}{2x} = \underline{\underline{1/4}}$$

$$2) \lim_{x \rightarrow 2} \frac{2x^2 + x - 6}{x+2} = \lim_{x \rightarrow 2} \frac{4x+1}{1} = \underline{\underline{-7}}$$

$$3) \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = \underline{\underline{0}}$$

$$4) \lim_{x \rightarrow \infty} \frac{5x^2 - 3x}{x^2 + 1} = \frac{10x - 3}{14x} = \frac{10}{14} = \underline{\underline{5/7}}$$

} here we can use previous method 2 if powers are same notice coefficient of highest power

$$\textcircled{Q} \lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = \underline{\underline{0}}$$

$$\textcircled{Q} \lim_{x \rightarrow 0} \frac{e^x - (1-x)}{x} = \frac{e^x + 1}{1} = \underline{\underline{2}}$$

$$\textcircled{Q} \lim_{x \rightarrow 1} \frac{x^{11} - 1}{x^4 - 1} = \lim_{x \rightarrow 1} \frac{11x^{10}}{4x^3} = \underline{\underline{11/4}}$$

$$\textcircled{Q} \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} = \lim_{x \rightarrow 0} 2e^{2x} = \underline{\underline{2}}$$

$$\textcircled{Q} \lim_{x \rightarrow 0} \frac{\sin 6x}{4x} = \lim_{x \rightarrow 0} \frac{6 \cos 6x}{4} = \underline{\underline{3/2}}$$

or $\lim_{x \rightarrow 0} \left(\frac{\sin 6x}{6x} \right) \cdot \frac{6x}{4x} \Rightarrow \frac{3}{2}$
→ standard

$$\textcircled{1} \lim_{x \rightarrow 1} \frac{\ln x^2}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{1 \cdot 2x}{x^2(2x)} = \frac{1}{x^2} = 1 //$$

$$\textcircled{2} \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} = \lim_{x \rightarrow \infty} 2 \ln x \cdot \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{2}{x} = 0 //$$

$$\textcircled{3} \lim_{x \rightarrow 0^+} \sqrt{x} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/\sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{1}{x^{-1/2}} \cdot \frac{-1}{2} x^{-3/2} = \frac{-2 \cdot \sqrt{x}}{x} = 0 //$$

How to solve indeterminate form of powers ∞^0 , 1^∞ , 0^0

$\lim_{x \rightarrow a} f(x)^{g(x)}$ is of the form ∞^0 , 1^∞ or 0^0

$$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x) \cdot \ln f(x)}$$

$$\Rightarrow \lim_{x \rightarrow a} f(x)^{g(x)} = \lim_{x \rightarrow a} e^{g(x) \cdot \ln f(x)} = e^{\lim_{x \rightarrow a} g(x) \cdot \ln f(x)}$$

by property 4 of Limits
 $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$
 if f is continuous

Here e^x is continuous

$$\text{so } \lim_{x \rightarrow a} e^{f(x)} = e^{\lim_{x \rightarrow a} f(x)}$$

→ given problem $\lim_{x \rightarrow a} f(x)^{g(x)}$

→ we first solve $\lim_{x \rightarrow a} g(x) \cdot \ln f(x)$

→ Then exponentate the result.

$$\begin{aligned} a &= e^{\ln a} \\ (e^a)^b &= (e^b)^a = e^{ab} \\ \ln(ab) &= b \cdot \ln a \\ \ln(m \cdot n) &= \ln(m) + \ln(n) \\ \ln\left(\frac{m}{n}\right) &= \ln(m) - \ln(n) \\ -\ln(n) &= \ln\left(\frac{1}{n}\right) \end{aligned}$$

$$\textcircled{1} \lim_{x \rightarrow 0^+} x^x = l \quad \lim_{x \rightarrow 0^+} x \cdot \ln x = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \frac{-1}{x^2} = \lim_{x \rightarrow 0^+} \frac{-1}{x^3} = 0$$

$$\therefore l = e^0 = 1 //$$

$$\textcircled{2} \lim_{x \rightarrow \infty} (e^x + x)^{1/x} = l \Rightarrow \log l = \lim_{x \rightarrow \infty} \ln \frac{(e^x + x)}{x} = \lim_{x \rightarrow \infty} \frac{1}{e^x + x} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x + 1} = 1 //$$

$$l \Rightarrow e^1 = e //$$

$$\begin{aligned} \textcircled{3} l &= \lim_{x \rightarrow 0^+} x^{\sin x} \Rightarrow \log l = \lim_{x \rightarrow 0^+} \sin x \cdot \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/\sin x} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \frac{1}{\frac{-1}{\sin^2 x}} \\ &= \lim_{x \rightarrow 0^+} \frac{-\sin^2 x}{x \cdot \cos x} = \lim_{x \rightarrow 0^+} \frac{-2 \sin x \cdot \cos x}{\cos x + x \sin x} = 0 // \end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \rightarrow \infty} x \cdot \ln\left(1 + \frac{1}{x}\right) = \log e = \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} \cdot \frac{-\frac{1}{x^2}}{-\frac{1}{x^2}} = 1 \Rightarrow e = e^1 = e$$

standard limit put $y = 1/x$

$$\lim_{y \rightarrow 0} \frac{\ln(y+1)}{y} = 1$$

Trick

$$\textcircled{1} \quad \lim_{\substack{f(x) \rightarrow 0 \\ g(x) \rightarrow \infty}} \left[1 + f(x)\right]^{g(x)} = \exp \left[\lim_{\substack{f(x) \rightarrow 0 \\ g(x) \rightarrow \infty}} \left[g(x) \cdot \ln(1 + f(x)) \right] \right]$$

$$= \exp \left[\lim_{\substack{f(x) \rightarrow 0 \\ g(x) \rightarrow \infty}} \left[g(x) \cdot \frac{\ln(1 + f(x))}{f(x)} \cdot f(x) \right] \right] = \exp \left[\lim_{\substack{f(x) \rightarrow 0 \\ g(x) \rightarrow \infty}} g(x) \cdot f(x) \right]$$

Standard $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$

general method can be applied to all questions involving 1^∞ .

a) Case $f(x) = 1/g(x)$ then $\lim_{f(x) \rightarrow 0} \left[1 + f(x)\right]^{\frac{1}{f(x)}} = e$

$$\Phi \quad \lim_{x \rightarrow 0} (1 + \sin x)^{1/\sin x} = \exp \left(\lim_{x \rightarrow 0} \frac{\ln(1 + \sin x)}{\sin x} \right) \xrightarrow[\sin x = y]{\text{COV}} \exp \lim_{y \rightarrow 0} \frac{\ln(1+y)}{y} = e //$$

$$\Phi \quad \lim_{x \rightarrow 0} (1+x)^{1/x} = \exp \left(\lim_{x \rightarrow 0} \left(\frac{1}{x} \cdot x \right) \right) = e^1 = e //$$

$$\Phi \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{3x} = \exp \left(\lim_{x \rightarrow \infty} 3x \cdot \frac{1}{x} \right) = e^3 //$$

$$\Phi \quad \lim_{x \rightarrow \infty} (1+x^2)^{1/x} = \exp \left(\lim_{x \rightarrow \infty} \frac{1}{x} \cdot x^2 \right) = e^0 = 1 //$$

$$\Phi \quad \lim_{x \rightarrow 0} (1-6x)^{1/x} = \exp \left(\lim_{x \rightarrow 0} \frac{1}{x} \cdot -6x \right) = e^{-6} //$$

$$\phi) \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx} = \exp \left(\lim_{x \rightarrow \infty} bx \cdot \frac{a}{x} \right) = e^{ab} //$$

gate CSE PYQ

$$\phi) \lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^{2x} = \exp \left(\lim_{x \rightarrow \infty} 2x \cdot \frac{3}{x} \right) = e^6 //$$

$$\phi) \lim_{n \rightarrow \infty} (n^2 - 1)^{1/n} = \exp \left(\lim_{n \rightarrow \infty} \frac{1}{n} \ln(n^2 - 1) \right) = \exp \left(\lim_{n \rightarrow \infty} \left(\frac{2n}{n^2 - 1} \right) \right) = \exp \left(\lim_{n \rightarrow \infty} \frac{2}{n} \right) = e^0 = 1 //$$

$$\phi) \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n} = e^2 //$$

$$\star \star \star \phi) \lim_{x \rightarrow a} f(x)^{g(x)} \text{ where } f(x) \rightarrow 1 \text{ \& } g(x) \rightarrow \infty$$

$$a) \lim_{x \rightarrow a} e^{g(x) \cdot f(x)}$$

$$b) \lim_{x \rightarrow a} e^{(g(x)-1) \cdot f(x)}$$

$$\checkmark c) \lim_{x \rightarrow a} e^{g(x) \cdot [f(x)-1]}$$

$$d) \lim_{x \rightarrow a} e^{(g(x)+1) \cdot (f(x)-1)}$$

$$\phi) \lim_{x \rightarrow 0} \frac{x(e^x - 1) + 2(\cos x - 1)}{x(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{(e^x - 1) + x \cdot e^x - 2 \sin x}{(1 - \cos x) + x \sin x} = \frac{e^x + e^x + x e^x - 2 \cos x}{1 + \sin x + \sin x + x \cos x}$$

$$\lim_{x \rightarrow 0} \frac{2e^x + e^x + x e^x + 2 \sin x}{2 \cos x - \sin x + \cos x} = \frac{3}{3} = 1 //$$

$$\star \phi) \lim_{x \rightarrow \infty} \left(\frac{x^2 + 5x + 3}{x^2 + x + 3} \right)^x = \exp \left(\lim_{x \rightarrow \infty} x \cdot \left(\frac{x^2 + 5x + 3}{x^2 + x + 3} - 1 \right) \right) = \exp \left(\lim_{x \rightarrow \infty} \frac{4x^2}{x^2 + x + 3} \right) = e^4 //$$

$$\phi) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{3x} = e^3 //$$

$$\phi) \lim_{x \rightarrow 0} (1 - 6x)^{1/x}$$

$$\phi) \boxed{\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x = e^k //$$

$$\Rightarrow e^{-6} //$$

$$\phi) \lim_{n \rightarrow \infty} (n^2 - 1)^{1/n} = \exp \left(\lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \ln(n^2 - 1) \right) \right) = \exp \left(\lim_{n \rightarrow \infty} \frac{1 \cdot 2n \cdot \frac{1}{n^2}}{n^2 - 1} \right) = e^0 = 1 //$$

$$\phi) \lim_{n \rightarrow \infty} (e^n + n)^{1/n} = \exp \left(\lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \ln(e^n + n) \right) \right) = \exp \left(\lim_{n \rightarrow \infty} \left(\frac{e^n + 1}{e^n + n} \right) \right) = \exp \left(\lim_{n \rightarrow \infty} \frac{e^n}{e^n + 1} \right) = e^1 //$$

$$\phi \quad \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n} = \lim_{n \rightarrow \infty} \frac{2 \ln(n)}{n} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0 //$$

$$\phi \quad \lim_{n \rightarrow \infty} \frac{3n^3 + 7}{e^n} = \lim_{n \rightarrow \infty} \frac{9n^2 + 1}{e^n} = \lim_{n \rightarrow \infty} \frac{18n}{e^n} = 0 //$$

$$\phi \quad \lim_{n \rightarrow \infty} \frac{n^2 - 5n}{e^n} = 0 //$$

$$\star \phi \quad \lim_{n \rightarrow \infty} \frac{n - \sin n}{n + \cos n} = \frac{n(1 - \frac{\sin n}{n})}{n(1 + \frac{\cos n}{n})} = 1 //$$

$$\phi \quad \lim_{x \rightarrow \infty} (x^{1/x}) = \exp\left(\lim_{x \rightarrow \infty} \frac{1}{x} \cdot \ln x\right) = \exp\left(\lim_{x \rightarrow \infty} \frac{1}{x}\right) = e^0 = 1 //$$

$$\phi \quad \lim_{x \rightarrow \infty} (1+x^2)e^{-x} = \exp\left(\lim_{x \rightarrow \infty} x^2 \cdot e^{-x}\right) = \exp\left(\lim_{x \rightarrow \infty} \frac{x^2}{e^x}\right) = e^0 = 1 //$$

$$\phi \quad \lim_{x \rightarrow 4} \frac{\sin(x-4)}{(x-4)} = \lim_{x \rightarrow 4} \frac{\cos(x-4)}{1} = 1 //$$

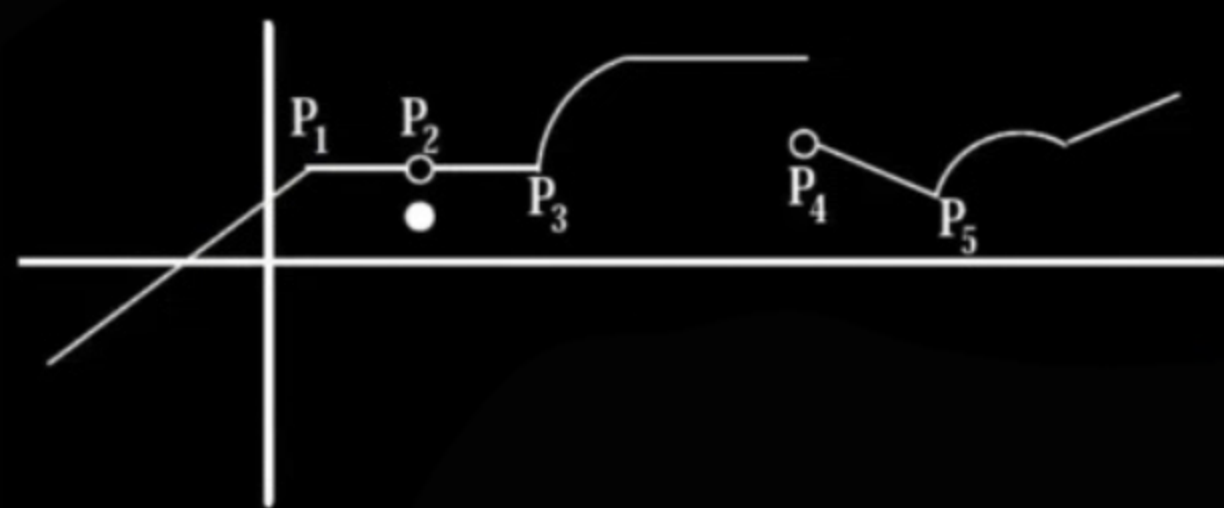
$$\phi \quad \lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim_{x \rightarrow 1} \frac{7x^6 - 10x^4}{3x^2 - 6x} = \frac{-3}{-3} = 1 //$$

$$\phi \quad \lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \rightarrow 3} \frac{4x^3}{4x - 5} = \frac{108}{7} //$$

$$\star \phi \quad \lim_{x \rightarrow 3} \frac{\sqrt{2x+22} - 4}{x+3} = \lim_{x \rightarrow 3} \frac{2x+22-16}{(x+3)(\sqrt{2x+22}+4)} = \frac{1}{8} \lim_{x \rightarrow 3} \frac{2x+6}{(x+3)} = \frac{1}{8} \lim_{x \rightarrow 3} \frac{2}{1} = \frac{1}{4} //$$

Definition of Continuity

Intuitively $\rightarrow$ Continuous at a point if we can graph be drawn through the point without lifting pencil.



$P_1, P_3 \neq P_5$

P_2, P_4 — not continuous

Formal definition: A function $y = f(x)$ is continuous at $x = a$ if $f(a)$ is defined and $\lim_{x \rightarrow a} f(x) = f(a)$

Consequences $\rightarrow$ Limit has to exist
 $\rightarrow LHL = RHL = f(a)$
 $\rightarrow f(a)$ should be defined

when this is not the case
 $\rightarrow$ Jump in function
 $\rightarrow f(a)$ is defined separately

Q) Determine if $f(x) = \frac{x^2 - 4}{x - 2}$ is continuous at $x = 2$.

No! $f(2)$ does not exist but $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = (x + 2) = 4$

Side note
 We can only cancel factors that become zero, inside the limit and not in function

ϕ $f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ not continuous @ $x = 0$

$\lim_{x \rightarrow 0} f(x) = 0 \neq f(0) = 1$ Limit $\neq f(a)$

ϕ Find all values of A | f is continuous in $(-\infty, \infty)$

$f(x) = \begin{cases} x + A & x < 2 \\ Ax^2 - 3 & x \geq 2 \end{cases}$

Solution

POI is $f(2) = 4A - 3$

$\lim_{x \rightarrow 2^-} f(x) = 2 + A$
 $\lim_{x \rightarrow 2^+} f(x) = 4A - 3$

Now $f(2^-) = 11/3$
 $f(2^+) = 11/3$

Q $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$ is $f(x)$ continuous at $x=2$

Sol
 $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = x+2 = 4 \neq f(2) = 3$
 $\lim \neq f(a) \Rightarrow$ not continuous @ $x=2$

Q) $f(x) = \begin{cases} x^2+3x+1 & \text{if } x < 1 \\ 2x^4+6x^2-3 & \text{if } x \geq 1 \end{cases}$ is $f(x)$ continuous @ $x=1$?

Sol
 $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2+3x+1 = 5$
 $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x^4+6x^2-3 = 2+6-3 = 5 = f(1)$
 $\therefore f(x)$ is continuous @ $x=1$

Q $g(x) = \begin{cases} x^2+3x+1 & \forall x < 1 \\ 7 & \forall x = 1 \\ 2x^4+6x^2-3 & x > 1 \end{cases}$ $\left. \begin{matrix} \text{prev} \\ 9 \end{matrix} \right\}$ LHL = RHL $\Rightarrow f(a) \therefore$ Not continuous

Q Determine where the fn. is not continuous.

$f(x) = \frac{x^2+x-6}{x^2+2x-3}$

Solution Trick $\rightarrow$ identity when denominator $\rightarrow 0$

$f(x) = \frac{(x^2+x-6)}{x^2+2x-3} = \frac{x^2+x-6}{(x+1)^2-2^2} = \frac{(x+3)(x-2)}{(x+3)(x-1)}$
 $\therefore$ Not continuous @ $x = \underline{\underline{-3, 1}}$

Remember we cannot cancel factors @ 0

Q $f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} & \forall -1 \leq x < 0 \\ \frac{2x+1}{x-1} & \forall 0 \leq x < 1 \end{cases}$

find k s.t. f is continuous @ $x=0$

Sol
 $f(0) = \lim_{x \rightarrow 0^+} f(x) = -1$

$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$

$= \lim_{x \rightarrow 0^-} \frac{(1+kx) - (1-kx)}{x(\sqrt{1+kx} + \sqrt{1-kx})} = \lim_{x \rightarrow 0^-} \frac{2k}{\sqrt{1+kx} + \sqrt{1-kx}} = \frac{2k}{2} = k$

$\therefore k$ has to be $\underline{\underline{-1}}$

$$f(x) = \begin{cases} 3ax+b, & x > 1 \\ 11, & x = 1 \\ 5ax-2b, & x < 1 \end{cases} \quad \text{find } a, b \text{ s.t. } f(x) \text{ is continuous @ } x=1$$

Solution

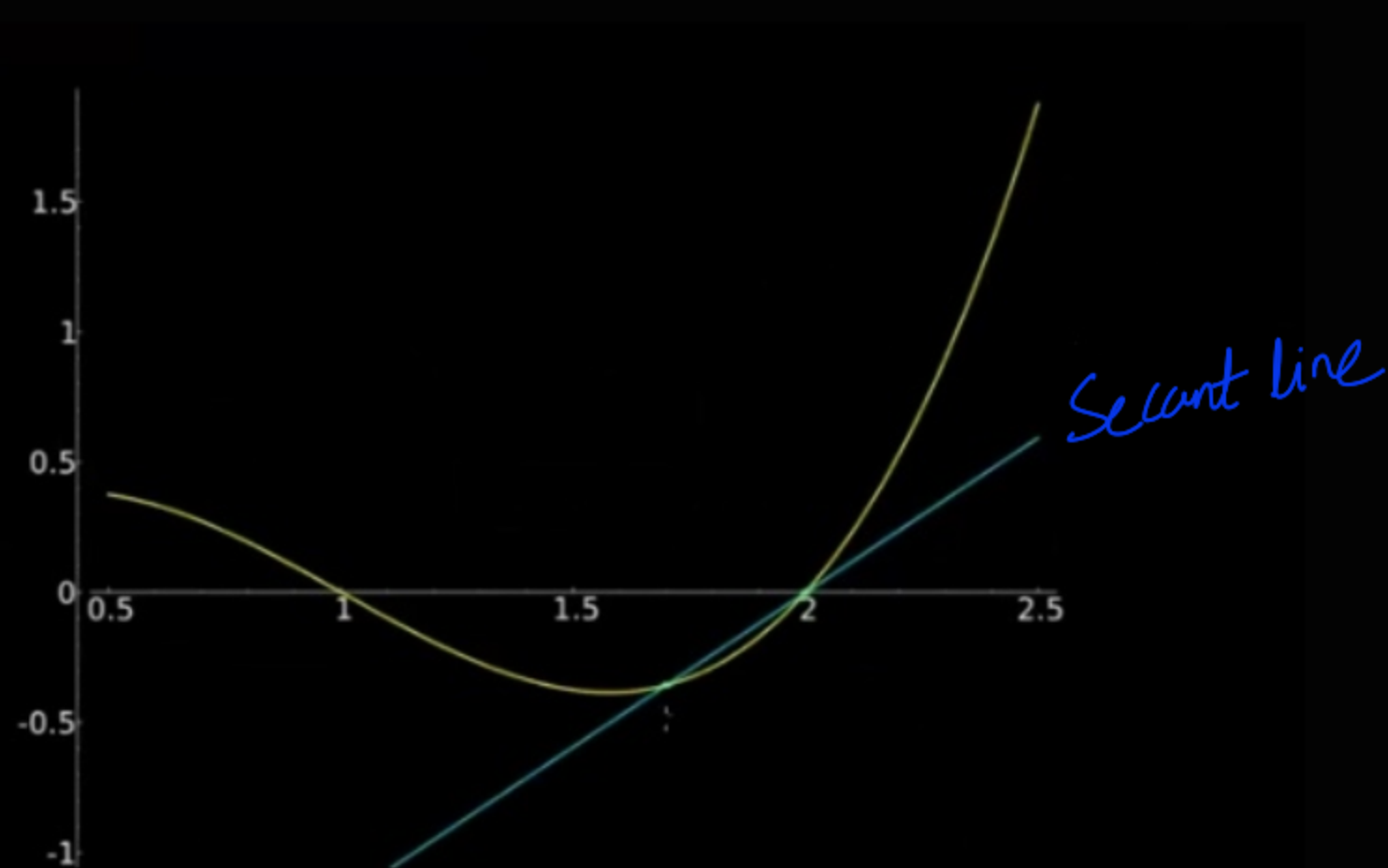
$$\lim_{x \rightarrow 1^-} f(x) = 5a - 2b; \quad \lim_{x \rightarrow 1^+} f(x) = 3a + b$$

2 eq. 2 unknowns.

$$\begin{bmatrix} 5 & -2 & | & 11 \\ 3 & 1 & | & 11 \end{bmatrix} \xrightarrow{R_1 \rightarrow 3R_1 - 5R_2} \begin{bmatrix} 0 & -11 & | & -22 \\ 3 & 1 & | & 11 \end{bmatrix} \Rightarrow \begin{matrix} b = 2 \\ a = 3 \end{matrix}$$

Function is continuous $\Rightarrow$ Limit exists
 Limit exists $\nRightarrow$ Function is continuous

DIFFERENTIABILITY



We are approximating the tangent line with secant lines

Let say points $P_0 = (a, f(a))$ & $P_1 = (b, f(b))$

(m) Slope of line $\rightarrow$ 2 point formula

$$m = \frac{f(b) - f(a)}{b - a}$$

$$m = \frac{f(a+h) - f(a)}{h}$$

where $b = a+h$

To approximate a tangent assume $b = a+h$ and $h \rightarrow 0$

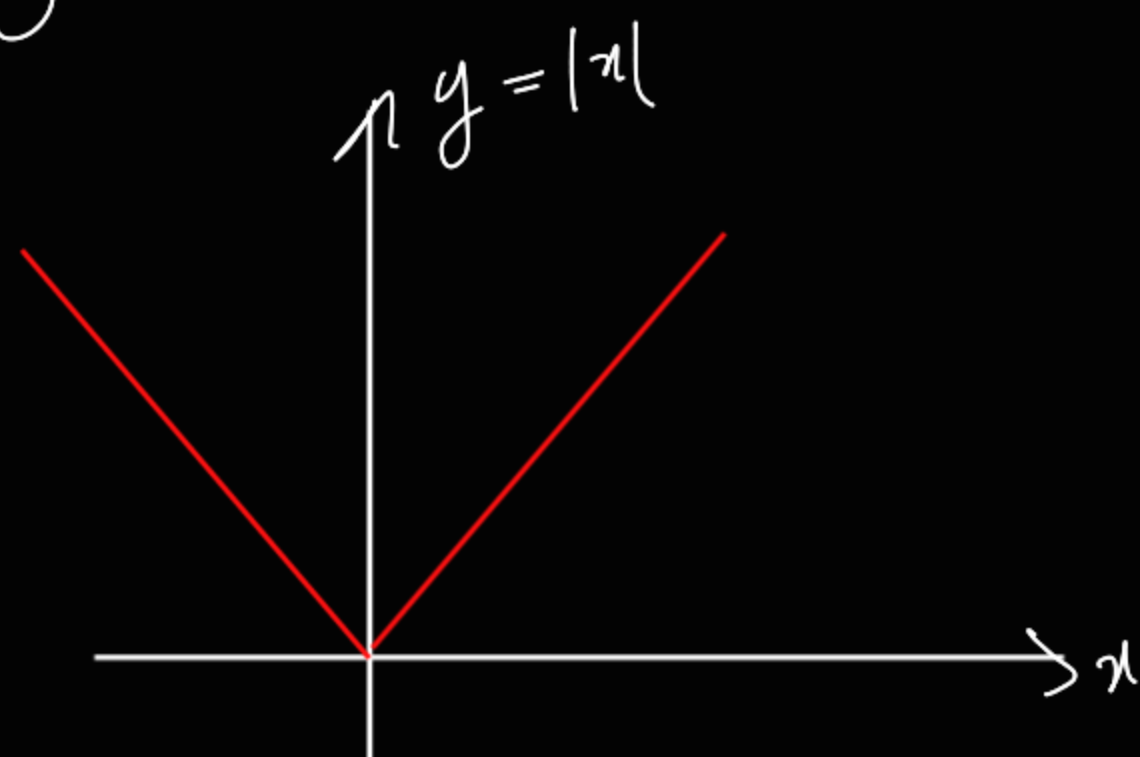
now the tangent line is the limiting case where $a \rightarrow b$ or $h \rightarrow 0$

$$\therefore \text{slope of tangent at point } a = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{RHL}$$

$$\text{Equivalent to } \lim_{h \rightarrow 0^-} \frac{f(a) - f(a-h)}{h} \quad \text{LHL}$$

$\rightarrow$ For a differentiable function, slope from left = slope from right

Q) $y = |x|$ is it differentiable @ $x=0$



in this example } right hand slope = +1
 @ $x=0$ } left hand slope = -1
 both not equal = Not diff @ $x=0$

Since $LHL \neq RHL$ the limit of slope does not exist at point where not differentiable.

FORMAL DEFINITION

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(a) - f(a+h)}{h}$$

Q Find $f'(-1)$ of $x^2 + 2x$ using formal definition (without standard limits)

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 + 2(x+h) - x^2 - 2x}{h} = \frac{x^2 + h^2 + 2xh + 2x + 2h - x^2 - 2x}{h} = \frac{h^2 + 2xh + 2h}{h}$$

$$f'(h) = \lim_{h \rightarrow 0} \frac{h(h + 2x + 2)}{h} = \lim_{h \rightarrow 0} h + 2x + 2 = 2x + 2$$

} This is the same result that we get using standard limit also

$$f'(-1) = 0 //$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, x > 0$$

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x \ln a}, x > 0$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

} standard derivatives
#sidequest

$$\text{Prove. } \ln_a x = \frac{\ln x}{\ln a}$$

$$\text{Let } x = a^m = (e^k)^m = e^{km}$$

$$\Rightarrow \ln x = km \quad \left\{ \begin{array}{l} \text{where} \\ e^k = a \\ \ln a = k \\ \ln_a x = m \end{array} \right.$$

CHAIN RULE

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$\text{eg: } \frac{d}{dx}(\sin x^3) = \cos(x^3) \cdot 3x^2$$

PRODUCT RULE

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

$$\text{eg: } \frac{d}{dx}[(x^2 + 3x + 5)(6x^3 - 7)] = (2x + 3)(6x^3 - 7) + 18x^2(x^2 + 3x + 5)$$

QUOTIENT RULE

$$\frac{Dd(n) - nD(d)}{d^2}$$

$$\frac{DDN - NDD}{D^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Denominator $\cdot$ Derivative of Numerator - Numerator $\cdot$ Derivative Denominator
Denominator²

$$d \frac{e^x - x^3 - 1}{2x} = \frac{2x(e^x - 3x^2) - 2(e^x - x^3 - 1)}{4x^2}$$

$$d \ln(\cos(x)) = \frac{1}{\cos x} \cdot -\sin x = \underline{\underline{-\tan x}}$$

$$d \frac{3^x + x^3}{x} \rightarrow a^x \ln a + 3x^2$$

★ Differentiation of piecewise function

→ In piecewise functions, the points of change in definition has to be checked if function is continuous, left derivative = right derivative & if differentiable, derivative @ these points have to be found explicitly.

$$\text{eg: } f(x) = \begin{cases} -x^2, & x \leq 0 \\ 0, & 0 < x < 1 \\ x-1, & x \geq 1 \end{cases} \Rightarrow f'(x) = \begin{cases} -2x, & x < 0 \\ a, & x = 0 \\ 0, & 0 < x < 1 \\ b, & x = 1 \\ 1, & x > 1 \end{cases}$$

$$a: f'(0^-) = 0 = f'(0^+) = 0 \Rightarrow f'(0) = 0$$

$$b: f'(1^-) = 0 \neq f'(1^+) = 1 \Rightarrow f(x) \text{ not differentiable @ } x=1$$

$$d f(x) = \begin{cases} 5-2x, & x < 0 \\ x^2-2x+5, & x \geq 0 \end{cases} \xrightarrow{\text{sol}} f'(x) = \begin{cases} -2, & x < 0 \\ -2, & x = 0 \\ 2x-2, & x > 0 \end{cases}$$

$$f(x) = \begin{cases} ax+b, & x \leq -1 \\ ax^3+x+2b, & x > -1 \end{cases} \xrightarrow{\text{sol}} f'(x) = \begin{cases} a, & x < -1 \\ 3ax^2+1, & x > -1 \end{cases}$$

also function It should exist @ $x=-1$

$$\Rightarrow -a+b = -a-1+2b$$

$$\Rightarrow b = 1 //$$

$$3a+1 = a$$

$$\underline{\underline{a = -1/2}}$$

Function is continuous $\Rightarrow$ Limit exists

Limit exists $\nRightarrow$ Function is continuous

Function is differentiable $\Rightarrow$ Function is continuous

$\Rightarrow$ Limit exist (reverse not true)

} At any point x_0

Maxima & Minima

- optimisation problem
- gradient
- Hessian
- Vector calculus

very important for gate DA

Taylor series of a function $f(x)$ about a point a_0

$$f(x+a_0) = \frac{f(a_0)}{0!} + \frac{x f'(a)}{1!} + \frac{x^2 f''(a)}{2!} + \frac{x^3 f'''(a)}{3!} + \dots + \frac{x^n f^{(n)}(a)}{n!}$$

$$f(x+a_0) = \sum_{n=0}^{\infty} \frac{x^n f^{(n)}(a)}{n!}$$

COV

$$f(x) = \frac{f(a_0)}{0!} + \frac{(x-a_0) f'(a_0)}{1!} + \frac{(x-a_0)^2 f''(a_0)}{2!} + \dots + \frac{(x-a_0)^n f^{(n)}(a_0)}{n!}$$

FIRST DERIVATIVE TEST

① $f'(a) > 0 \rightarrow$ function is increasing @ point a

② $f'(x) > 0 \mid x \in (a,b) \rightarrow$ function is increasing on interval (a,b)

③ $f'(a) < 0 \rightarrow$ function is decreasing @ point a

④ $f'(x) < 0 \mid x \in (a,b) \rightarrow$ function is decreasing on interval (a,b)

⑤ $f'(x) = 0$, then x is either

① local minima: $f'(x) \xrightarrow{-ve \text{ to } +ve} x$

② local maxima: $f'(x) \xrightarrow{+ve \text{ to } -ve} x$

③ stationary point: $f'(x) \xrightarrow{+ve \ 0 \ -ve}{-ve \ 0 \ -ve} x$

} critical points

SECOND DERIVATIVE TEST any $x \in (a,b)$, then if

① $f''(x) > 0 \rightarrow$ slope of function increasing $\rightarrow$ Concave up / Convex function in (a,b)

② $f''(x) < 0 \rightarrow$ slope of function decreasing $\rightarrow$ Concave down / Concave fn in (a,b)

③ $f''(x) = 0 \rightarrow$ slope is not changing $\rightarrow f''(x)$ changes sign $\rightarrow$ inflection
 $\rightarrow f'(x) = 0$
stationary inflection

local maxima local minima

1. if $f(x)$ is continuous in (a,b) & $f'(a) = 0$ & $f''(a) > 0$ for some $a \in (a,b)$
then $f(x)$ has a critical point at $f(a)$ with increasing slope $\rightarrow$ local minima.
2. if $f(x)$ is continuous (a,b) & $f'(a) = 0$ & $f''(a) < 0$ for some $a \in (a,b)$
then $f(x)$ has a critical point at $f(a)$ with decreasing slope $\rightarrow$ local maxima
3. if $f(x)$ is continuous (a,b) & $f'(a) = 0$ & $f''(a) = 0$ for some $a \in (a,b)$
then $f(x)$ has a critical point at $f(a)$ with critical slope $\rightarrow$ Stationary inflection point

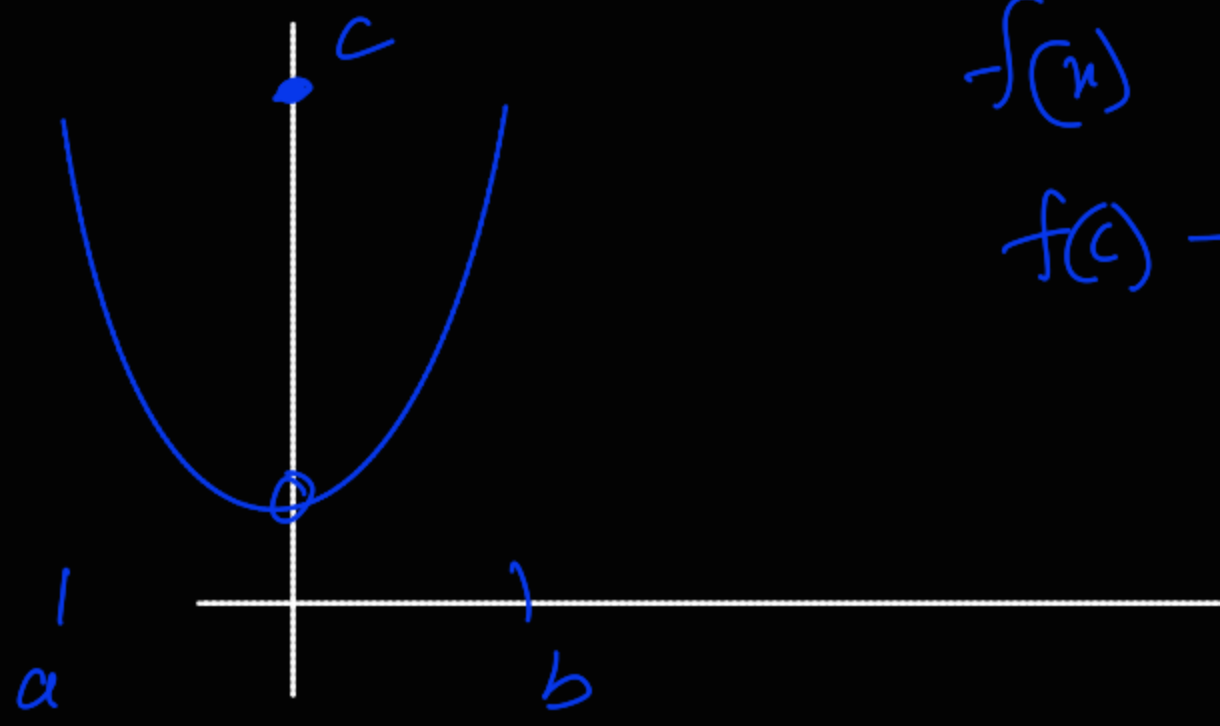
If f has a local maximum or minimum at c , then $f'(c) = 0$.

False $\rightarrow f'(c)$ could not be defined or c could be edge of function.

If $f'(c) = 0$, then $f(c)$ is either a local maximum or a local minimum.

False $\rightarrow$ could be a saddle point

eg.



$f(x)$ domain (a, b)

$f(c) \rightarrow \text{maxima} \nRightarrow f'(c) = 0$

Q $f(x) = 2x^3 + x^2 - 8x + 2$ find absolute maxima & absolute minima

f on $[0, 2]$ solution

$$f'(x) = 6x^2 + 2x - 8 = 0 \Rightarrow 3x^2 + x - 4 = 0$$

$$(3x+4)(x-1) = 0$$

$$f(1) = 2 + 1 - 8 + 2 = -3$$

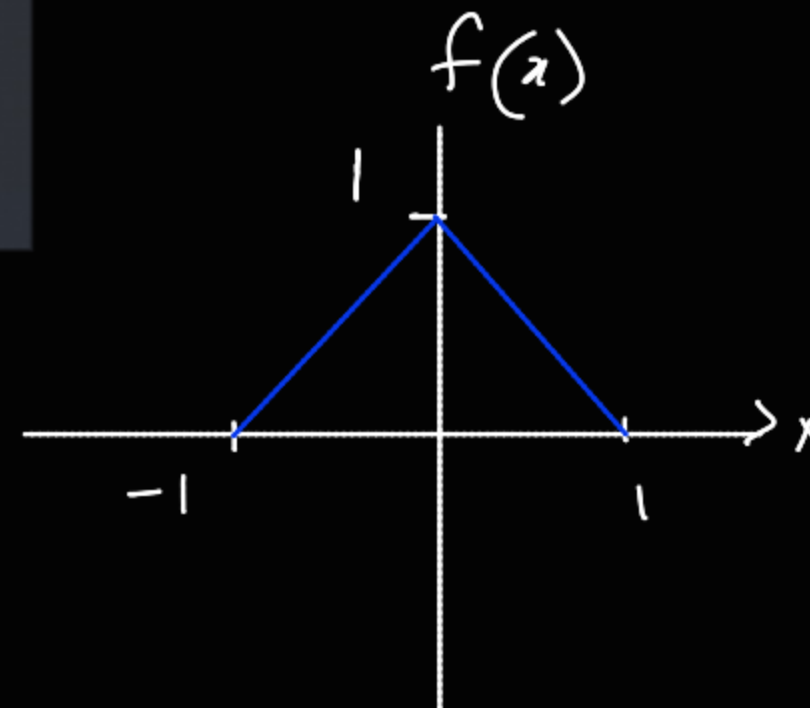
$$f(0) = 2$$

$$f(2) = 16 + 4 - 16 + 2 = 6$$

$$f(x) = (3x+4)(x-1)$$

Consider a function $f(x) = 1 - |x|$ on $-1 \leq x \leq 1$. The value of x at which the function attains a maximum, and the maximum value of the function are:

- A. 0, -1
- B. -1, 0
- C. 0, 1
- D. -1, 2



#sidequest

$$ax^2 + bx + c = 0$$

$$= x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

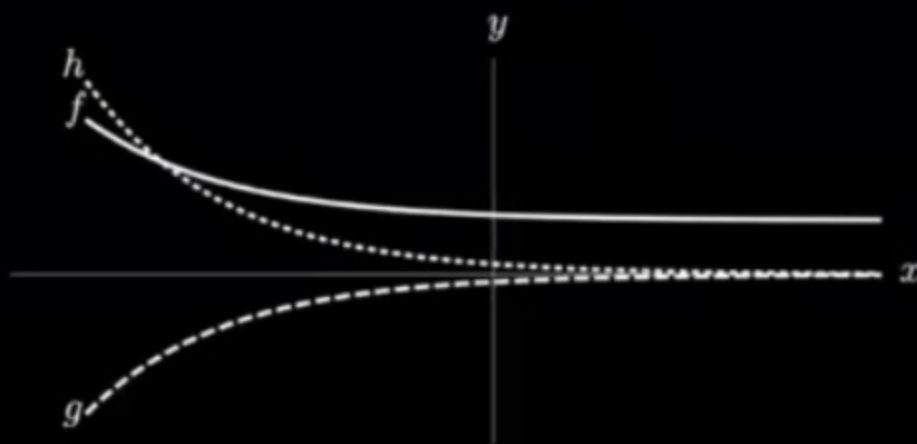
$$\Rightarrow x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = 0$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{\pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

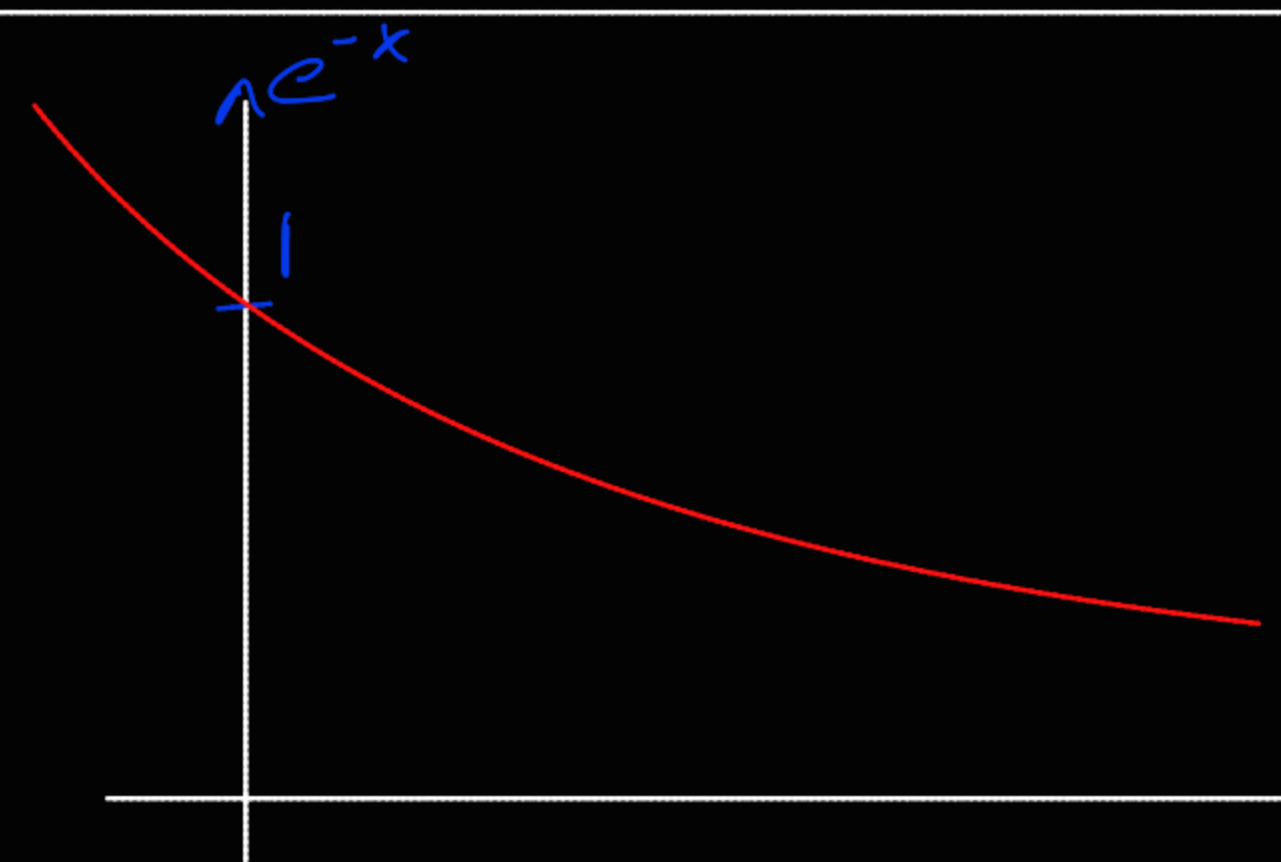
Which of the following statement is true for the given graphs of 3 functions?



- (a) h is the derivative of f , and f is the derivative of g . X
- (b) g is the derivative of f , and f is the derivative of h . X
- (c) g is the derivative of h , and h is the derivative of f . X
- (d) h is the derivative of g , and g is the derivative of f . ✓

(j) There exists a function f such that $f(x) > 0$ and $f'(x) < 0$ and $f''(x) > 0$ for all x .

function = e^{-x} or $1/x$



$$\phi \quad f(x) = 3x^4 - 8x^3 - 6x^2 + 24x - 7$$

$$f'(x) = 12x^3 - 24x^2 - 12x + 24 = 0$$

a) Intervals of Inc & dec

$$f''(x) = 36x^2 - 48x - 12 = 0$$

$$3x^2 - 4x - 1 = 0$$

$f(x)$ decreasing in $(-\infty, -1)$

$f(x)$ increasing in $(-1, 1)$

$f(x)$ decreasing in $(1, 2)$

$f(x)$ increasing in $(2, \infty)$

$$f'(x) = 0 \Rightarrow x^3 - 2x^2 - x + 2 = 0$$

$$x = 1, -1, 2$$

$$x = \frac{4 \pm \sqrt{16 + 12}}{6} = \frac{4 \pm \sqrt{28}}{6}$$

$$\left(\frac{2 \pm \sqrt{7}}{3} \right) = \frac{2 \pm \sqrt{7}}{3}$$

$$(x-1)(x+1) = x^2 - 1$$

$$\begin{array}{r} x^3 - 2x^2 - x + 2 \\ \underline{x^3 - x^2 - 1} \\ x^2 - x + 2 \\ \underline{x^2 - x - 1} \\ 3 \end{array}$$

b) $f''(-1) = (3 + 4 - 1)12 = 72$ local mini

$f''(1) = (3 - 4 - 1)12 = -24$ local max

$f''(2) = (12 - 8 - 1)12 = 36$ local min

c) $f'(x) = 0$ @ $\frac{2 \pm \sqrt{7}}{3}$

$$\therefore f''\left(\frac{2}{3}\right) = 12 \left[3\left(\frac{4}{9}\right) - 4\left(\frac{2}{3}\right) - 1 \right]$$

$$= 12 \left[\frac{4}{3} - \frac{8}{3} - \frac{3}{3} \right] = -24$$

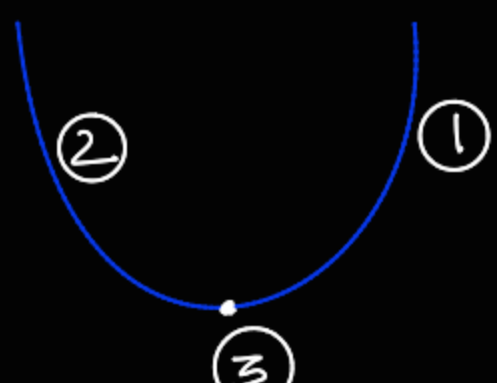
between $\left(\frac{2 - \sqrt{7}}{3}, \frac{2 + \sqrt{7}}{3} \right)$

$f''(x) < 0 \therefore$ Concave down

SECOND DERIVATIVE TEST

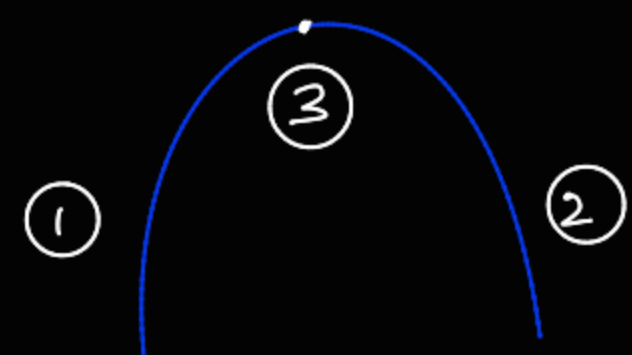
☆ $f''(x) > 0 \rightarrow$

① $f'(x) > 0 \rightarrow$ Increasing @ increasing rate	} Concave up Convex
② $f'(x) < 0 \rightarrow$ decreasing @ increasing rate	
③ $f'(x) = 0 \rightarrow$ Minima point - local	



☆ $f''(x) < 0 \rightarrow$

① $f'(x) > 0 \rightarrow$ Increasing @ decreasing rate	} Concave down Concave
② $f'(x) < 0 \rightarrow$ decreasing @ decreasing rate	
③ $f'(x) = 0 \rightarrow$ Maxima point - local	



☆ $f''(x) = 0 \rightarrow$

① $f'(x) = 0 \rightarrow$ stationary point
② $f''(x)$ changes sign from left to right $\Rightarrow$ point of inflection

φ Find all extrema's of the following function

$$f(x) = 2x^5 - 3x^4 - 12x + 5$$

$$f'(x) = 10x^4 - 12x^3 - 12 = 0$$

$$6(x^2 - x - 2) = 0$$

$$6(x-2)(x+1) = 0$$

$$x = 2, -1$$

$$f''(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$

$$f''(2) = 18$$

$$f''(-1) = -18$$

$$f(2) = 16 - 12 - 24 + 5 = -15$$

$$f(2) = -15 = \underline{\text{Minima}}$$

$$f(-1) = -2 - 3 + 12 + 5 = 12 \text{ (Maxima)}$$

Find inflation $y = x^3 + x$

$$f'(x) = 3x^2 + 1 \quad f'(0) = 1$$

$$f''(x) = 6x = 0 \text{ @ } x=0$$

also changes sign $\therefore$ Inflation point

ORDINARY POINT OF INFLATION

$y = x^3 - 3x^2 + 3x - 1$

$$y' = 3x^2 - 6x + 3 \quad y'(1) = 3 - 6 + 3 = 0$$

$$y'' = 6x - 6$$

$$y''(1) = 0 \quad y'' > 0 \quad \forall x > 1$$

$$y'' < 0 \quad \forall x < 1$$

STATIONARY POINT OF INFLATION

ϕ given
 $f(x)$
is continuous
& differentiable

$f'(x)$	3	4	3	2	-1	-7	-2	4	6
x	-4	-3	-2	-1	0	1	2	3	4
$f''(x)$	+	-	-	-	-	-	+	+	+

a) $f(x)$ has local min btw 2 & 3

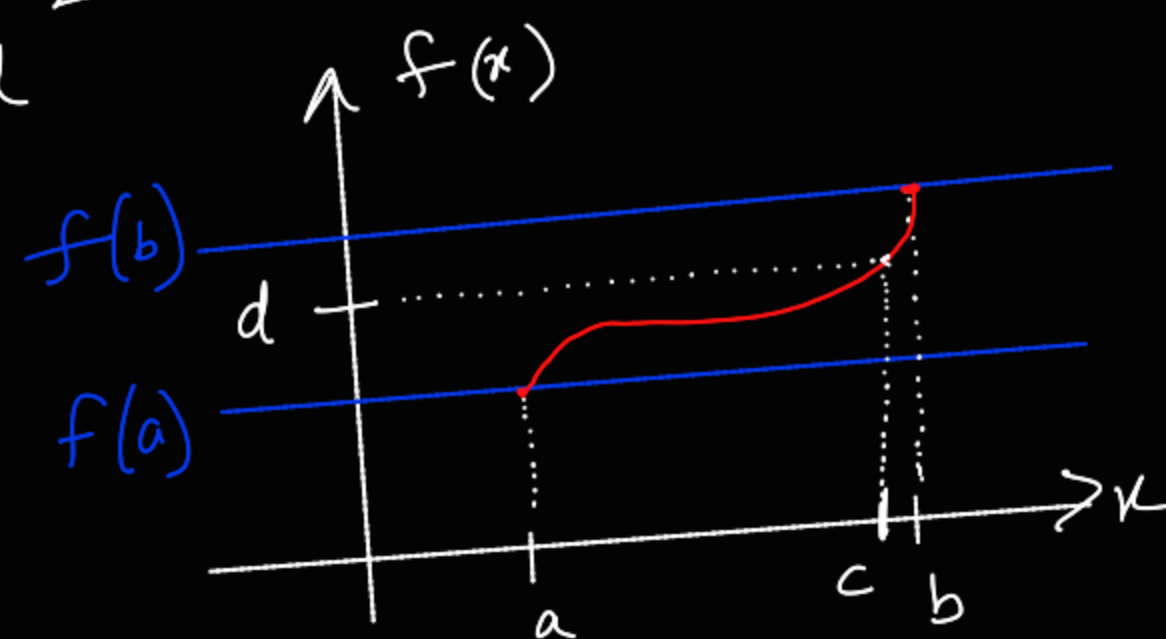
b) $f(x)$ has local max btw -1 & 0

c) $f(x)$ has inflection pt btw -4 & -2 (more than 1 possible)

Insight if $f(x)$ is continuous & differentiable

INTERMEDIATE VALUE THEOREM (IVT)

☆ if f is an continuous function on the closed interval $[a, b]$ & if d is between $f(a)$ & $f(b)$ then exist atleast one real $c \in [a, b]$ with $f(c) = d$



☆ $f(a)f(b) < 0 \Rightarrow f(a)$ & $f(b)$ are having opposite signs.

$\Rightarrow$ They will cut the axis

$\Rightarrow$ There exist atleast one real c such that $f(c) = 0$

ϕ Prove $y = x^4 - x - 4$ has root in interval $[-2, 2]$

$$\text{This is continuous } y(2) = 16 - 2 - 4 = 10 \quad y(-2) = 16 + 2 - 4 = 14$$

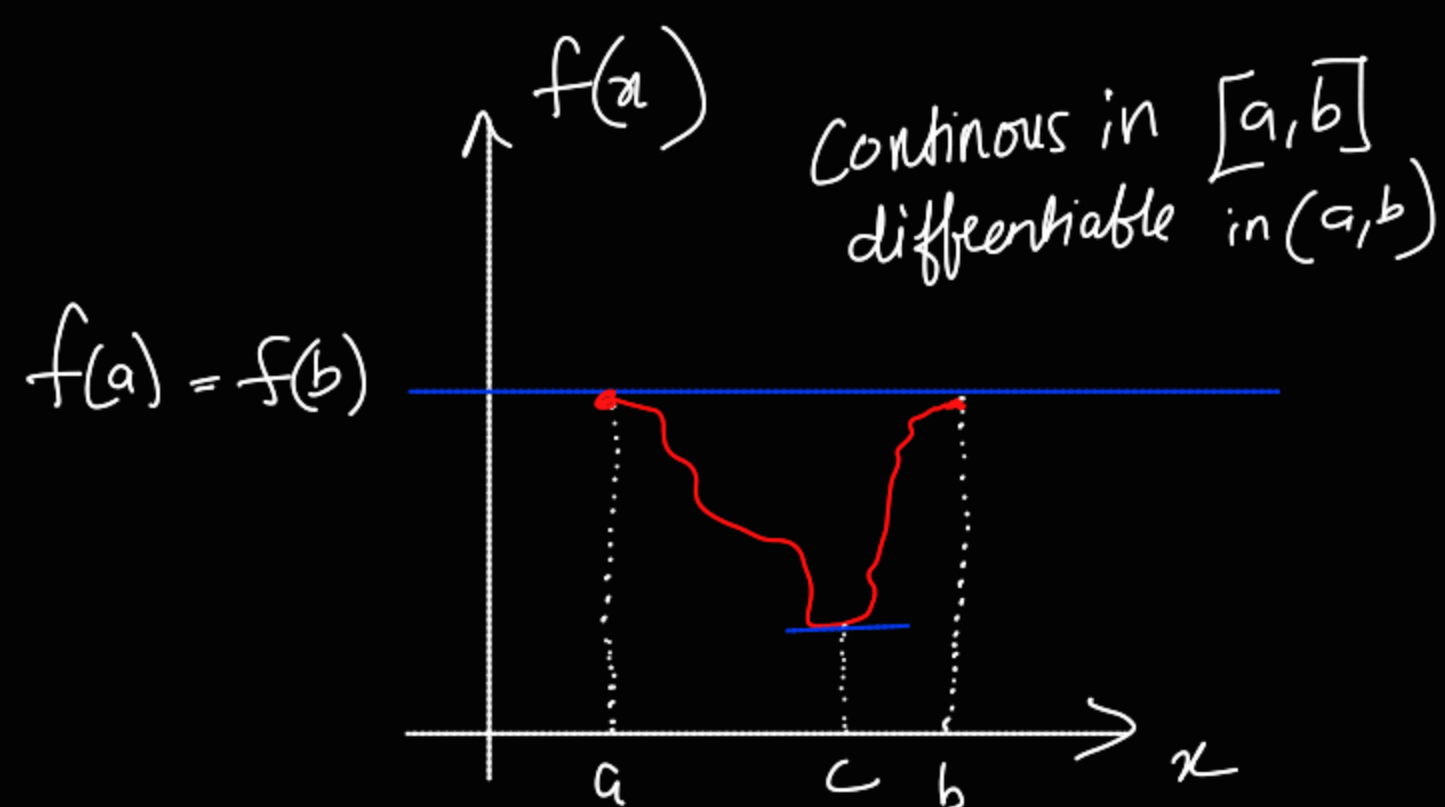
$$y(0) = -4 \quad \therefore \text{There are atleast 2 roots in } [-2, 2]$$

$\phi f(x) = \begin{cases} 0, & x \leq 0 \\ 1, & \text{otherwise} \end{cases} \quad \left. \begin{matrix} f(-1) = 0 \\ f(2) = 1 \end{matrix} \right\} f \rightarrow \text{never takes } 0.5$
why does it contradict IVT

Because it is not continuous, it doesn't follow IVT

ROLLE'S THEOREM

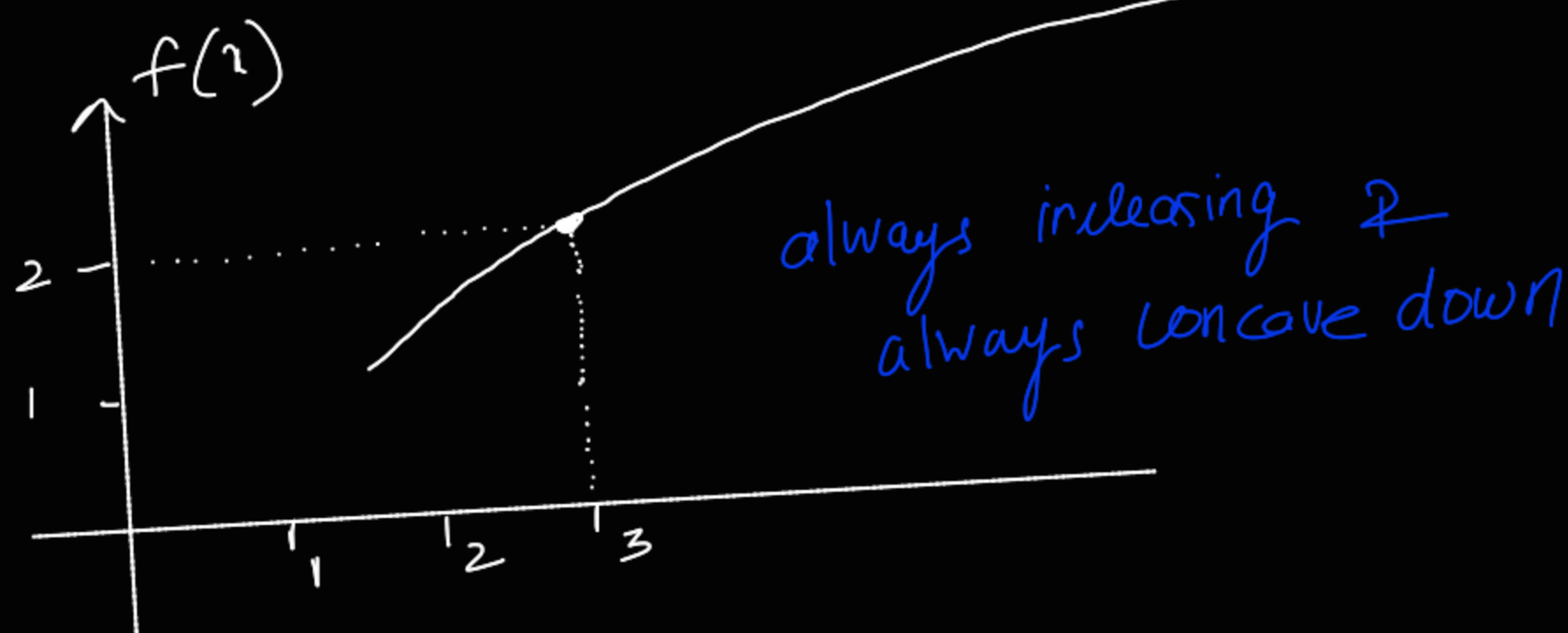
$f(x)$ is continuous on $[a, b]$ & differentiable on (a, b) , then if $f(a) = f(b)$ then there is at least one real $c \in [a, b]$ with $f'(c) = 0$



☐ $f(3) = 2, f'(3) = 1/2$ & $f'(x) > 0$ & $f''(x) < 0 \forall x$.

- Sketch f
- How many roots does f have?

→ exactly one



☐ $y = 2x + e^x \rightarrow$ show exactly 1 root

$y' = 2 + e^x > 0 \forall x \therefore$ By Rolle's theorem if root exist then exactly one.

$y(-1) = -2 + \frac{1}{e} < 0$
 $y(0) = 2 > 0$ } $\Rightarrow$ y is continuous in $[-1, 1]$ } $\therefore$ By IVT root exist

☆ IVT can be used to show root exist

☆ Rolle's theorem can be used to show only one root exist

☆ Rolle's theorem can be used to show a function doesn't repeat values

☆ To show properties of a function eg: $f(c) = f(c+1)$ for some $c \in [a, b]$ put $g(x) = \text{property}$
 eg $g(x) = f(x) - f(x+1)$ & show $g(x)$ has root in the interval using IVT

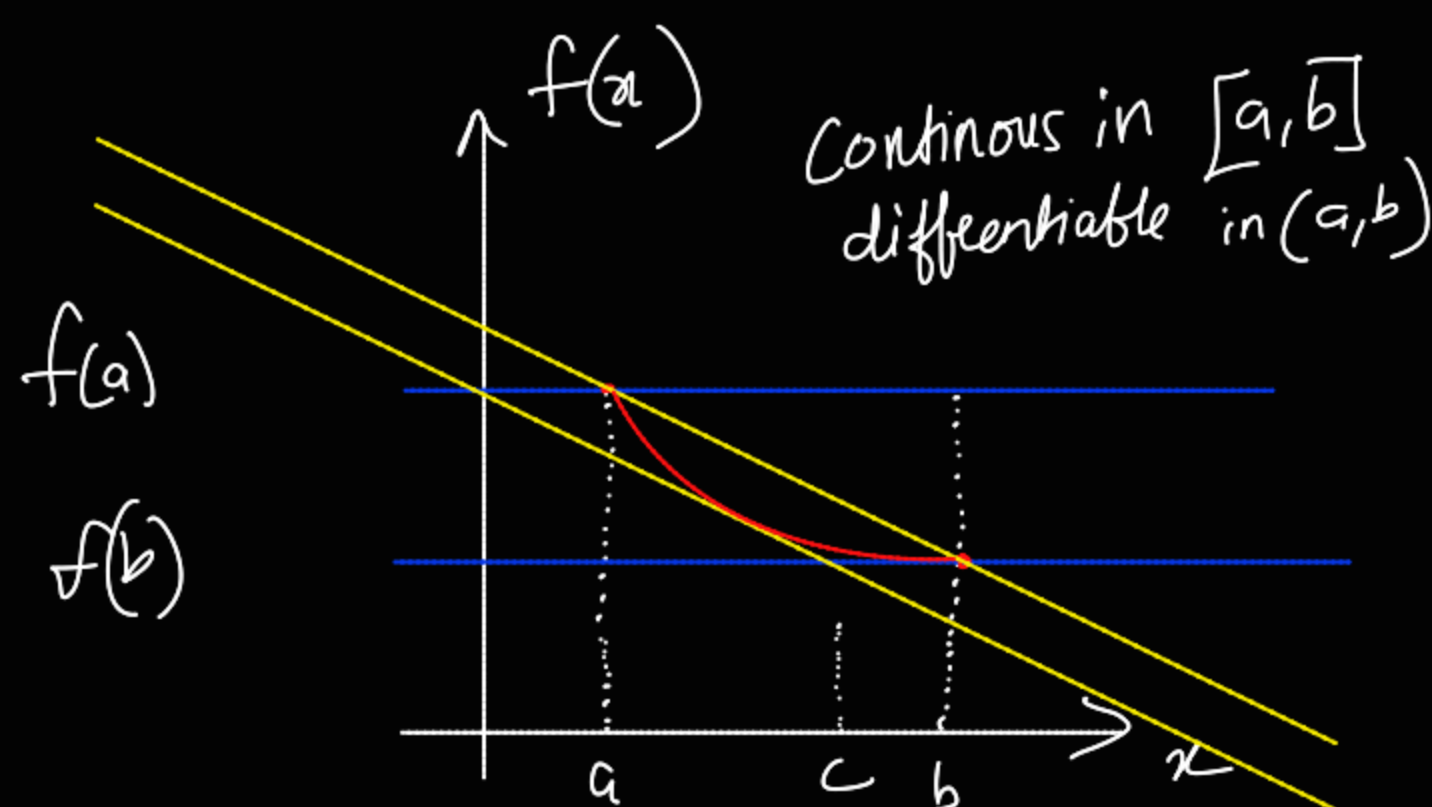
Mean Value Theorem if $f(x)$ continuous in $[a, b]$ & differentiable in $(a, b) \Rightarrow \exists$ at least one real $c \in (a, b)$

Lagrange's Theorem

such that, $f'(c) = \frac{f(b) - f(a)}{b - a}$

Generalisation of Rolle's theorem: if $f(b) = f(a)$

→ Average speed intuition — You have to travel at your average speed at least for 1 fraction of time.



ϕ T/F $\rightarrow \exists f(x) \mid f(x)$ is differentiable $\nexists f(0)=0 \nexists f(2)=2 \nexists f'(x) < 1 \forall x$
 false $\rightarrow$ By MVT $\exists c \in (0,2) \mid f'(c) = \frac{2-0}{2-0} = 1$
 This contradicts question

ϕ By using MVT, show that $\forall n \in \mathbb{I}^+ \rightarrow \sqrt{n+1} - \sqrt{n} = \frac{1}{2\sqrt{c}}$, for some $c \in (n, n+1)$

Solution let $f(x) = \sqrt{x} \rightarrow$ continuous & differentiable in $(0, \infty)$

then for any pos. integer n

$$\begin{aligned} f(n+1) &= \sqrt{n+1} \\ f(n) &= \sqrt{n} \end{aligned}$$
 $\rightarrow$ By MVT $\exists c \in (n, n+1) \mid f'(c) = \frac{\sqrt{n+1} - \sqrt{n}}{n+1 - n}$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \sqrt{n+1} - \sqrt{n}$$

ϕ T/F $\rightarrow f$ is differentiable & has distinct roots $a_1 < a_2$ then f' has at least 1 root $b \in (a_1, a_2)$
 True by MVT or Rolle's theorem

ϕ Gate CSE 2021 Set 2 Q25:

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $[-3, 3]$ & diffen in $(-3, 3)$ such that for every x in this interval $f'(x) \leq 2$. If $f(-3) = 7$, then $f(3)$ is at most.

Solution

$$\begin{aligned} f(3) &= f(-3) + f'(x) \cdot 6 \\ &= 7 + 2 \cdot 6 \\ f(3) &= 19 \end{aligned}$$
 by MVT

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - 7}{3 - (-3)}$$

$$\Rightarrow 7 + 2 \cdot 6 = f(3)$$

ϕ if 2 numbers add upto k what is largest their product can be.

Sol

$$\begin{aligned} x + y &= k & xy &= x(k-x) & f'(x) &= k - 2x = 0 \Rightarrow x = \frac{k}{2} \\ y &= k - x & f(x) = xy &= kx - x^2 \end{aligned}$$

$$\Rightarrow xy = \frac{k^2}{4}$$

if two are no odd so 12 then minimise their L_2 norm

$$x^2 + y^2 = \text{minimise} \quad \Rightarrow \quad x^2 + k^2 + x^2 - 2kx = f(x) = 2x^2 + k^2 - 2kx$$

$$x+y=k \quad \Rightarrow \quad f'(x) = 4x - 2k = 0 \Rightarrow x = k/2 = y$$

$$\Rightarrow x^2 + y^2 \Big|_{\text{minimum if } x+y=k} = \frac{k^2}{4} + \frac{k^2}{4} = \frac{k^2}{2}$$

☆☆
Important
Question
using IVT

Let $f: [0,2] \rightarrow \mathbb{R}$ is continuous with $f(0)=f(2)$. show $\exists c \in [0,1]$ with $f(c)=f(c+1)$

approach show $g(c) = f(c) - f(c+1)$ has opposite sign in $g(0) \neq g(1)$

$$g(0) = f(0) - f(1) \quad \text{But } f(0)=f(2)$$

$$g(1) = f(1) - f(2) \Rightarrow g(0) \neq g(1) \text{ must have opp sign}$$

$$\Rightarrow \exists c \in [0,1] \mid g(c) = 0 \Rightarrow$$

$$\exists c \in [0,1] \mid f(c) = f(c+1)$$

Gate CSE 2014

A fn $f(x)$ is continuous in $[0,2]$. It is known that $f(0)=f(2)=-1$ & $f(1)=1$.

Then which of the following is true?

- ✓ a) $\exists y \in (0,1)$ such that $f(y) = f(y+1)$
- b) $\forall y \in (0,1) \quad f(y) = f(2-y)$
- c) The maximum value of the fn in the interval $(0,2) = 1$
- ✓ d) $\exists y \in [0,1] \mid f(y) = -f(2-y)$

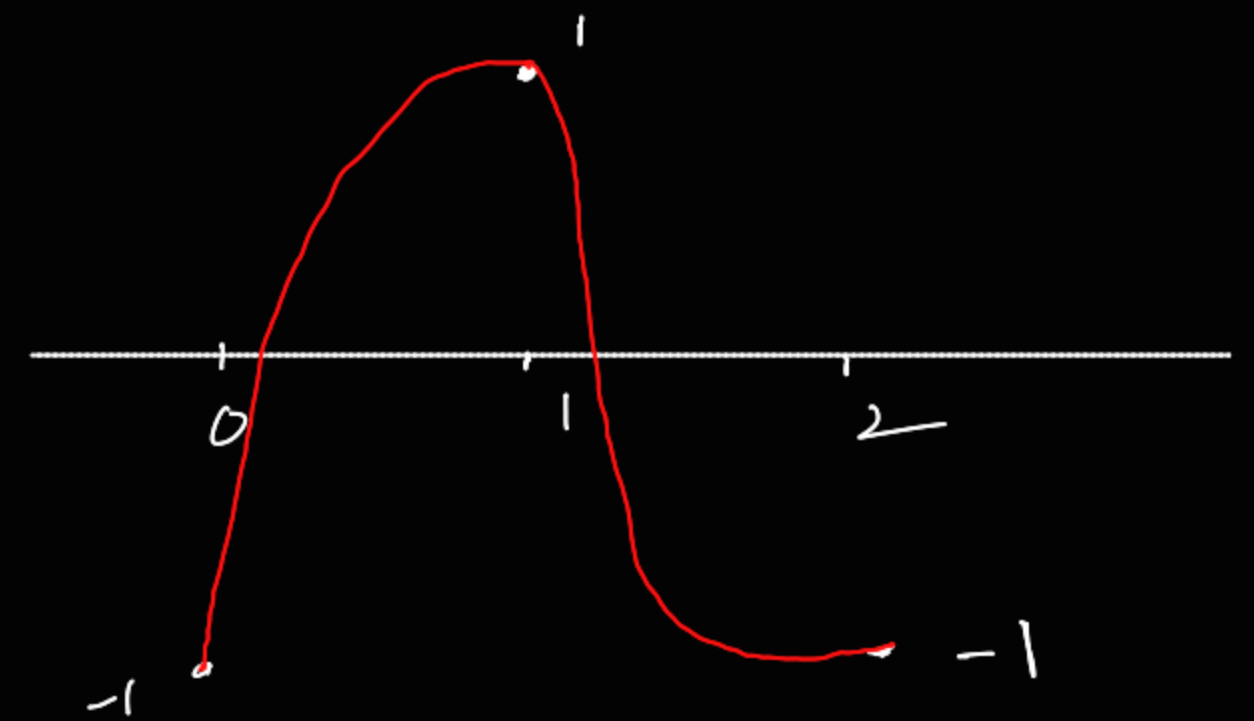
$$\rightarrow g(x) = f(y) + f(2-y)$$

$$g(0) = f(0) + f(2) = -2$$

$$g(1) = f(1) + f(1) = 2$$

$$\Rightarrow g(c) = 0 \text{ for some } c \in [0,1] - [0,1]$$

$$\Rightarrow f(y) = f(2-y) \text{ for some } y \in (0,1) \quad \text{since we know } g(0), g(1) \neq 0$$



Q No. of roots of $e^x + 0.5x^2 - 2$ in $(-5, 5)$

a 0
b 1
c 2
d 3

$$f(x) = e^x + 0.5x^2 - 2$$

$$\begin{aligned} f(-5) &> 0 \\ f(5) &> 0 \\ f(0) &< 0 \end{aligned} \Rightarrow \text{At least 2 solutions}$$

$$\begin{aligned} f'(x) &= 0 \Rightarrow e^x = -x \\ x &= \ln(-x) \text{ --- only one solution} \end{aligned}$$

$\Rightarrow$ Exactly 2 roots.

further $f''(x) = e^x$
 $\rightarrow f(x)$ increasing $\Rightarrow$ 1 solution for $f'(x) = 0$



P.T. $f(x) = x^3 - 3x + c$ has at most 1 root in $[0, 1]$ No matter c

$$f(0) = c$$

$$f(1) = 1 - 3 + c = c - 2$$

At least one root for $0 < c < 2$

$f(x)$ = polynomial
 $\therefore$ continuous &
diff $\in [0, 1]$

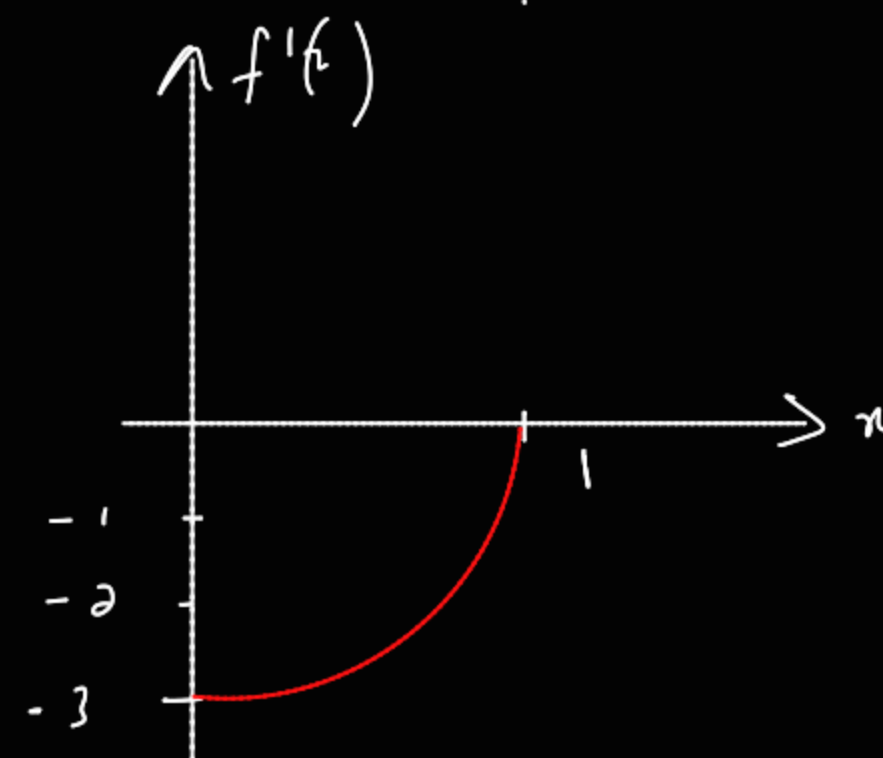
$$f'(x) = 3x^2 - 3$$

$$f'(x) < 0 \quad \forall x \in [0, 1]$$

$$f''(x) = 6x \geq 0 \quad \forall x \in [0, 1]$$

$$f'(0) = -3$$

$$f'(1) = 0$$



INTEGRATION

Standard Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int e^{ax} = \frac{e^{ax}}{a} + c$$

$$\int a^x = \frac{a^x}{\ln a} + c$$

$$\int \sin x \cdot dx = -\cos x + c$$

$$\int \cos x \cdot dx = \sin x + c$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + c$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\oint \frac{\cos(\ln x)}{x} \cdot dx \quad \text{put } y = \ln x$$

$$dy = \frac{1}{x} dx$$

$$\int \cos y \cdot dy$$

$$= \sin y + C = \sin(\ln x)$$

★ Change of Variable $\rightarrow$ Back substitution

$$\int_a^b \frac{f(g(x)) g'(x)}{g'(x)} \cdot dx \quad \text{put } g(x) = y$$

$$g'(x) \cdot dx = dy$$

$$\equiv \int_{g(a)}^{g(b)} f(y) \cdot dy$$

indefinite integral
back substitution needed
after integration

definite Integral - No back substitution just
change limits of integr

$$\oint \frac{dx}{\sqrt{x} (2 - \sqrt{x})}$$

$$\text{put } \sqrt{x} = y$$

$$dy = \frac{1}{2\sqrt{x}} dx$$

$$\equiv 2 \int \frac{1}{2-y} dy = -2 \ln(2-\sqrt{x}) + C$$

$$\oint \frac{x^3}{1+x^4} dx = \frac{1}{4} \tan^{-1} x^4 + C$$

$$\oint \frac{e^x}{4+e^{2x}} dx = \frac{1}{2} \tan^{-1} \left(\frac{e^x}{2} \right)$$

$$\oint \frac{1}{4+9x^2} dx = \frac{1}{6} \tan^{-1} \left(\frac{3x}{2} \right)$$

★ 2 Dividing by coefficient

$$\int_a^b f(kx) dx$$

$$\text{put } kx = t$$

$$k dx = dt$$

$$\equiv \frac{1}{k} \int_{ka}^{kb} f(t) \cdot dt$$

★ 2' Take -ve outside

using earlier result

$$\int_a^b f(-x) \cdot dx \equiv - \int_{-a}^{-b} f(t) \cdot dt$$

put $t = -x$

$$\oint \frac{\ln(\ln x)}{x \ln x} dx = \frac{(\ln(\ln x))^2}{2}$$

$$\oint x^3 \cdot e^{x^4} = \frac{1}{4} e^{x^4}$$

$$\oint \frac{1+x}{1+x^2} dx = \tan^{-1}(x) + \frac{1}{2} \ln(1+x^2)$$

★ 3 Splitting fractions

$$\int \frac{f(x) + g(x)}{h(x)} dx = \int \frac{f(x)}{h(x)} dx + \int \frac{g(x)}{h(x)} dx$$

$$\oint (3^x + 3^{-x} + 2) dx = \frac{3^x}{\ln(3)} - \frac{3^{-x}}{\ln(3)} + 2x$$

PROPERTIES OF DEFINITE INTEGRAL

$$(4) \int_a^b f(x) \cdot dx = \int_a^b f(t) dt \quad | \quad t=x$$

$$(5) \int_a^b f(x) \cdot dx = - \int_b^a f(x) \cdot dx$$

$$(6) \int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_c^b f(x) \cdot dx$$

$$(7) \int_a^b f(x) \cdot dx = \int_a^b f(a+b-x) \cdot dx \quad \star \star$$

put $a+b-x=t$
 $-dx=dt$
 $-\int_b^a f(t) dt$
 $= \int_a^b f(t) dt = \int_a^b f(x) \cdot dx$
 @ $x=a, t=b$
 @ $x=b, t=a$

Q gate CS 2024: Let $f(x)$ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x) = 1 - f(2-x)$$

which of following is correct value of $\int_0^2 f(x) dx$?

- a) 0
- b) 1
- c) 2
- d) -1

solution
 $I = \int_0^2 1 - f(2-x) dx = [x]_0^2 - \int_0^2 f(2-x) dx$

$$I = 2 + \int_2^0 f(t) \cdot dt =$$

$$2I = 2 \Rightarrow I = 1$$

$2-x=t$
 $-dx=dt$
 @ $x=0, t=2$
 @ $x=2, t=0$

gate shortcut $\rightarrow$ In questions where they give a property for a function/matrix or any other mathematical object without explicitly defining the value of the object. And then if they ask any other property, numerical or attribute / T/F about this object, one strategy is to assume the simplest example of function/matrix/math object that they defined & check the answer

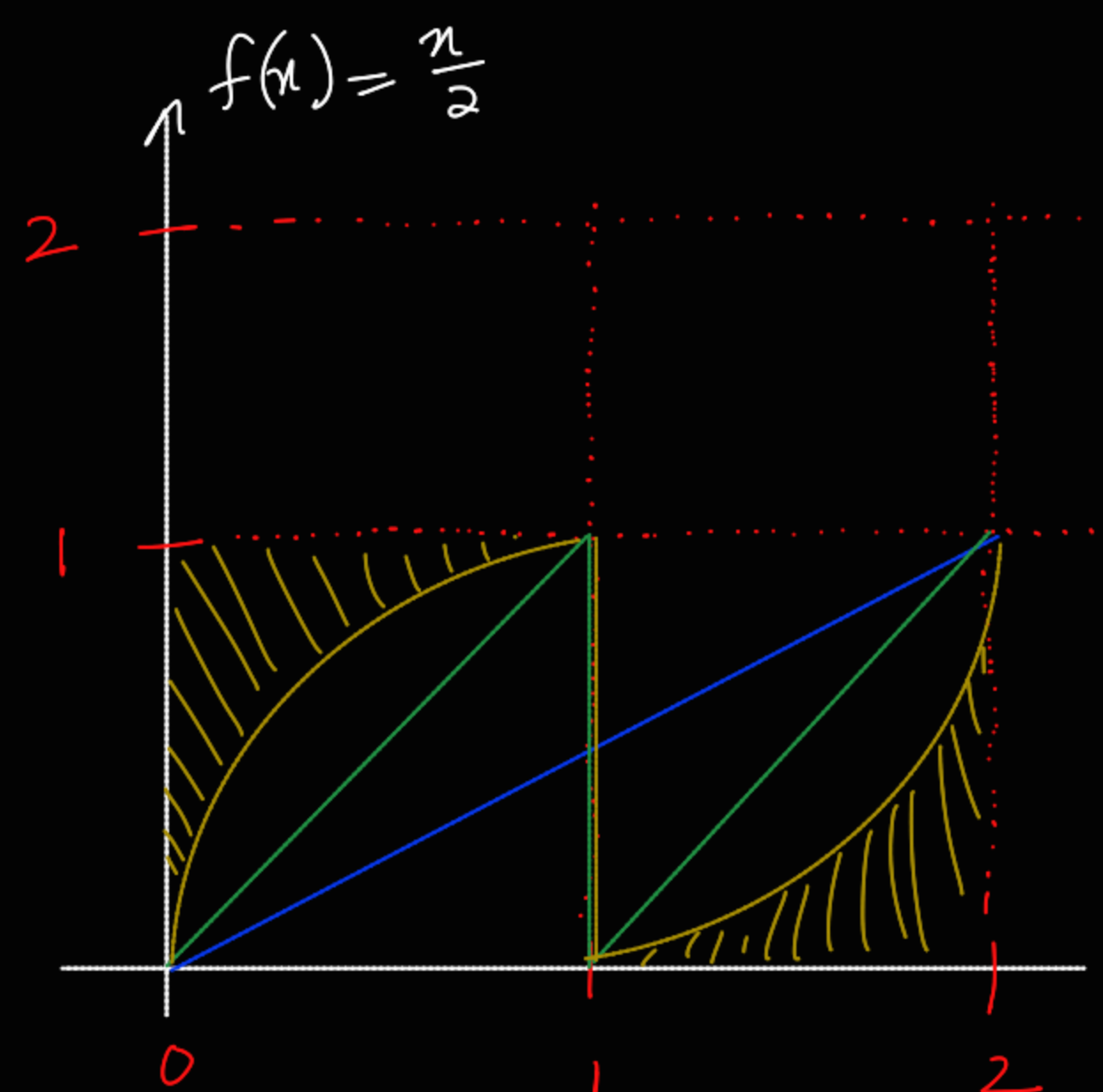
In the above question $f(x) = 1 - f(2-x)$

$$f(x) = \frac{1}{2}x$$

$$= 1 - \left(\frac{2-x}{2} \right)$$

$$\int_0^2 f(x) dx = \text{area} = 2 \times 1 \times \frac{1}{2} = 1$$

$$= \frac{2 - 2 + x}{2} = \frac{x}{2}$$



$$\oint \int_{50}^{100} \frac{\ln x}{\ln x \cdot \ln(150-x)} \cdot dx$$

$$2I = \int_{50}^{100} \frac{\ln x + \ln(150-x)}{\ln(150-x) + \ln(x)} dx = 50$$

$$I = \underline{\underline{25}}$$

without property.

$$\begin{aligned} & \int_{50}^{75} \frac{\ln x}{\ln x \cdot \ln(150-x)} + \int_{75}^{100} \frac{\ln x}{\ln x \cdot \ln(150-x)} \\ & \int_{50}^{75} \frac{\ln x}{\ln x \cdot \ln(150-x)} - \int_{75}^{50} \frac{\ln(150-y)}{\ln y \cdot \ln(150-y)} \\ & = \int_{50}^{75} \frac{\ln x}{\ln x \cdot \ln(150-x)} + \int_{50}^{75} \frac{\ln(150-y)}{\ln y \cdot \ln(150-y)} \\ & = \int_{50}^{75} 1 dx = \underline{\underline{25}} \end{aligned}$$

$$\oint \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx = \underline{\underline{1/2}}$$

$$\oint \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx = \int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx \quad \left. \begin{array}{l} \text{Using property 4} \\ \text{of integration} \\ \text{no path ahead.} \end{array} \right\}$$

put $\cos x = u$
 $-\sin x dx = du$

$$- \int_1^0 \frac{du}{1+u^2} dx = \left. \tan^{-1} u \right|_0^1 = \underline{\underline{\frac{\pi}{4} - 0}}$$

#Sidequest

$$\begin{aligned} \sin(\pi-x) &= \sin x \\ \sin(\pi+x) &= -\sin x \\ \cos(\pi-x) &= -\cos x \\ \cos(\pi+x) &= -\cos x \end{aligned}$$

Sin +	Sin +	Sin = 0
cos -	cos +	
Sin -	Sin -	cos = 0
cos -	cos +	

$$\begin{aligned} \sin(-x) &= -\sin x \\ \cos(-x) &= \cos x \\ \sin(\pi/2-x) &= \cos x \\ \sin(\pi/2+x) &= \cos x \\ \cos(\pi/2-x) &= \sin x \\ \cos(\pi/2+x) &= -\sin x \end{aligned}$$

$$\oint \int_e^\infty \frac{1}{x \ln^3 x} \cdot dx = -\frac{1}{2} \left[\frac{1}{u^2} \right]_1^\infty = \underline{\underline{\frac{1}{2}}}$$

ϕ If $f(x)$ is integrable and $\int_0^4 f(x) dx = 10$, $\int_2^4 f(x) dx = 6$, what is $\int_0^1 f(2x) dx$?

$$= \frac{1}{2} \int_0^2 f(x) \cdot dx = \frac{10-6}{2} = \underline{\underline{2}}$$

$$\oint \int_{-1}^3 \sqrt{1+x} \cdot dx$$

$$\begin{aligned} \sqrt{1+x} &= u \\ \frac{1}{2\sqrt{1+x}} dx &= du \end{aligned}$$

$$\Rightarrow 2 \int_0^2 u^2 du = \left. \frac{2}{3} u^3 \right|_0^2 = \underline{\underline{\frac{16}{3}}}$$

★ 1.1 $\int_a^b f(g(x)) dx = \int_a^b \frac{f(g(x)) g'(x)}{g'(x)} dx$ if we can put $\frac{f(g(x))}{g'(x)} = h(u)$ | put $g(x) = u$
 $g'(x) dx = du$
 $dx = \frac{du}{g'(x)}$
 $= \int_{g(a)}^{g(b)} h(u) du$

☐ $\int_1^4 \frac{10^{\sqrt{x}}}{\sqrt{x}} dx = \frac{2}{\ln(10)} \left[10^u \right]_1^2 = \frac{180}{\ln(10)}$ | ☐ $\int_1^9 \frac{5^{\sqrt{x}}}{\sqrt{x}} dx = \frac{2(25-5)}{\ln 5} = \frac{20}{\ln(5)}$

☐ $\int_0^{\pi/2} \cos^5 x \cdot dx$ put $\cos^2 x = u$
 $-2 \cos x \cdot \sin x dx = du$ $\Rightarrow -\frac{1}{2} \int_0^{\pi/2} \frac{(\cos^2 x)^2}{\sin x} \cdot 2 \cos x \cdot \sin x dx$

$= -\frac{1}{2} \int_1^0 \frac{u^2}{\sqrt{1-u}} du \rightarrow$ Not getting anywhere.

$I = \int_0^{\pi/2} \sin^5(x) dx \Rightarrow 2I = \int_0^{\pi/2} \sin^5(u) + \cos^5(x) \cdot dx \rightarrow$ Not getting anywhere.

$u = \sin x \Rightarrow du = \cos x \cdot dx$

★ $\int_0^{\pi/2} (1 - \sin^2 x)^2 \cdot \cos x \cdot dx = \int_0^1 (1-u^2)^2 \cdot du = \left[\frac{u^5}{5} - \frac{2u^3}{3} + u \right]_0^1 = \frac{1}{5} - \frac{2}{3} + 1 = \frac{15-10+3}{15} = \frac{8}{15}$
 ★★★★★★

INTEGRATION BY PARTS

$\int uv dx = u \int v dx - \int \left[\frac{du}{dx} \cdot \int v dx \right] dx$

★ 8. $\int u v' dx = uv - \int u' v dx$

PROOF $\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

$\frac{f(x)g(x)}{u \cdot v} = \frac{f'(x)}{u'} \cdot \frac{g(x)}{v} + \frac{f(x)}{u} \cdot \frac{g'(x)}{v'}$

$\Rightarrow \int u v' dx = uv - \int u' v$

Assume 2nd function which is easier to integrate

→ Use ILATE rule to choose the first & second function

Inverse Trig
 Logarithm
 Algebraic / polynomial
 Trigonometric
 Exponential
 easy

eg:- $\int x^3 e^{2x} dx$
 ILATE
 1st 2nd

$$\phi \int x^3 e^{2x} dx = x^3 \frac{1}{2} e^{2x} - \frac{1}{2} \int 3x^2 \cdot e^{2x} = \frac{x^3 e^{2x}}{2} - \frac{3}{2} \left[x^2 \cdot \frac{1}{2} e^{2x} - \frac{1}{2} \int 2x e^{2x} \right]$$

$$= \frac{x^3 e^{2x}}{2} - \frac{3}{4} x^2 e^{2x} + \frac{3}{2} \left[\frac{x e^{2x}}{2} - \frac{e^{2x}}{2 \cdot 2} \right] = \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3x e^{2x}}{4} - \frac{3e^{2x}}{8}$$

★ 8.1 $\int f(x) \cdot e^{ax} dx$ where $f(x) = \text{polynomial}$

$$= \frac{f(x)e^{ax}}{a} - \frac{f'(x)e^{ax}}{a^2} + \frac{f''(x)e^{ax}}{a^3} - \frac{f'''(x)e^{ax}}{a^4} \dots$$

TIC TAC TOE METHOD

until derivative is 1

→ same as above method.

★ 8.2 $\int f(x) \cdot e^{ax} \cdot dx$

$f(x) = \text{Trig}(x)$

→ Apply TIC TAC TOE until it becomes chicken & egg (self definition)

★ 8.3 $\int f(x) \cdot \text{Trig}(x) \cdot dx$
 $f(x) \rightarrow \text{poly}(x)$

	<u>Diff</u>	<u>Integ</u>
$+ x^3$		$\rightarrow e^{2x}/2$
$- 3x^2$		$\rightarrow e^{2x}/4$
$+ 6x$		$\rightarrow \frac{e^{2x}}{8}$
$- 6$		$\rightarrow e^{2x}/16$
$+ 0$		

★ $\int \ln x \, dx = x \cdot \ln x - \int \frac{1}{x} x \, dx = x \cdot \ln x - x = \underline{\underline{x(\ln(x) - 1)}}$

Q

If $f(1) = 2, f(4) = 7, f'(1) = 5, f'(4) = 3$, and $f''(x)$ is continuous, what is $\int_1^4 x f''(x) \, dx$?

a) 9

c) 12

e) 2



b) -2

d) -1

f) None of the above.

$$= \left[x f'(x) - f(x) \right]_1^4 = \frac{4 \cdot 3 - 7}{-(1 \cdot 5 - 2)}$$

$$= 12 - 7 - 5 + 2 = 2 //$$

Q $\int e^{2x} \sin x \, dx = \sin x \cdot \frac{e^{2x}}{2} - \cos x \frac{e^{2x}}{4} + \int -\sin x \cdot \frac{e^{2x}}{4} \, dx$

$$I = \frac{\sin x e^{2x}}{2} - \frac{\cos x e^{2x}}{4} - \frac{I}{4}$$

$$I = \frac{4}{5} \left(\frac{\sin x e^{2x}}{2} - \frac{\cos x e^{2x}}{4} \right)$$

8.14 $\int e^{ax} \cdot \sin bx \cdot dx = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx) - b \cos(bx)]$

$\int e^{ax} \cos bx \cdot dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx) + b \sin(bx)]$

$$\int e^{ax} \cos(bx) \cdot dx = \frac{\cos(bx) \cdot e^{ax}}{a} + \frac{b}{a} \int \sin bx \cdot e^{ax} \cdot dx = \frac{\cos bx \cdot e^{ax}}{a} + \frac{b}{a^2} \sin bx \cdot e^{ax} - \frac{b^2}{a^2} \int \cos bx \cdot e^{ax}$$

$$\Rightarrow I \left(1 + \frac{b^2}{a^2}\right) = e^{ax} \left(a \frac{\cos bx}{a^2} + \frac{b \sin bx}{a^2} \right) \Rightarrow I = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

$$\oint \int x^5 \ln x \cdot dx \quad \ln x \cdot \frac{x^6}{6} - \int \frac{1}{x} \cdot \frac{x^6}{6} = \frac{1}{6} \left(x^6 \cdot \ln(x) - \frac{x^6}{6} \right) = \underline{\underline{\frac{x^6}{6} (\ln(x) - \frac{1}{6})}}$$

$\oint \int \ln(x)^2 \cdot dx = \int u^2 e^u \cdot du$

$\ln(x) = u \quad x = e^u$
 $\frac{1}{x} dx = du$
 $du = \frac{1}{x} dx$
 $dx = e^u \cdot du$

$\Rightarrow u^2 e^u - 2u e^u + 2e^u$
 $= e^u (u^2 - 2u + 2)$
 $= \underline{\underline{x (\ln^2 x - 2 \ln x + 2)}}$

Substitution + Integration by parts.

$\star \int_0^{\pi/2} \sin^3(x) \cdot \cos^2(x) \cdot dx = \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1 = \frac{1}{3} - \frac{1}{5} = \underline{\underline{\frac{2}{15}}}$

$\cos x = u$
 $-\sin x \cdot dx = du$
 $\int_0^1 u^2 (1 - u^2) du$
 $\text{@ } x=0 \quad u=1$
 $\text{@ } x=\pi/2 \quad u=0$

$$\oint \int_{-\infty}^0 e^x \sin x \cdot dx = \frac{e^x}{2} [1 \cdot \sin x - 1 \cdot \cos x]_{-\infty}^0 = \frac{1}{2} [0 - 1] - 0 = \underline{\underline{-1/2}}$$

$$\oint \int_0^{\infty} e^{-x} \cos x \cdot dx = \frac{e^{-x}}{2} [-\cos x + \sin x]_0^{\infty} = \frac{1}{2} [1] = \underline{\underline{1/2}}$$

★ GATE CSE 2009

$$\int_0^{\pi/4} (1 - \tan x)/(1 + \tan x) dx$$

- A. 0
B. 1
C. $\ln 2$
✓ D. $1/2 \ln 2$

$$\tan\left(\frac{\pi}{4} - x\right) = \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} = \frac{1 - \tan x}{1 + \tan x}$$

$$\begin{aligned} &= \int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right) dx = \int_0^{\pi/4} \tan x \cdot dx = \left(\ln |\cos x|\right)_0^{\pi/4} \\ &= \ln \left|\frac{1}{\sqrt{2}}\right| - \ln 1 \\ &= \underline{\underline{\frac{1}{2} \ln 2}} \end{aligned}$$

$$\begin{aligned} \frac{\sin x}{\cos x} & \quad \cos x = u \\ \sin x dx &= du \\ &= \frac{1}{u} du = \ln(u) \\ &= \ln |\cos x| + C \end{aligned}$$

GATE CSE 2015

Compute the value of:

$$\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\cos(1/x)}{x^2} dx$$

$$\begin{aligned} \text{put } \frac{1}{x} &= u & @ x = \frac{1}{\pi} \quad u = \pi \\ -\frac{1}{x^2} dx &= du & @ x = \frac{2}{\pi} \quad u = \frac{\pi}{2} \\ \int_{\pi/2}^{\pi} \cos u (du) &= \sin \pi - \sin \pi/2 \\ &= \underline{\underline{-1/2}} \end{aligned}$$

GATE IT 2005

What is the value of $\int_0^{2\pi} (x - \pi)^2 (\sin x) dx$

- A. -1
B. 0
C. 1
D. π

$$\sin(\pi + u) = -\sin u$$

$$\begin{aligned} \text{put } x - \pi &= u \\ dx &= du \\ x &= \pi + u \end{aligned}$$

$$= - \int_{-\pi}^{\pi} u^2 \cdot \sin u \, du$$

$$\begin{aligned} &= + \left(u^2 - \cos u - du(-\sin u) + 2 \cos u \right)_{-\pi}^{\pi} \\ &= \pi^2 - \cos(\pi) - (\pi^2 - \cos \pi) + 2 \cos(\pi) - 2 \cos \pi \\ &= \pi^2(1-1) + 2(-1+1) \\ &= \underline{\underline{0}} \end{aligned}$$

GATE CSE 2011

Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral $\int_0^{\pi/2} \frac{\cos x + i \sin x}{\cos x - i \sin x} dx$?

- A. 0
B. 2
C. $-i$
✓ D. i

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned}$$

$$= \int_0^{\pi/2} e^{i2x} dx = \frac{e^{i2x}}{i2} \Big|_0^{\pi/2}$$

$$\begin{aligned} &= \frac{\cos 2x + i \sin 2x}{i2} \Big|_0^{\pi/2} = \frac{-1 + i0}{2i} - \frac{1 + i0}{2i} \\ &= \frac{-1}{i} = i \end{aligned}$$

$$\int \frac{-5x^4}{(1-x^2)^{5/2}} dx = -5 \int \frac{x^4}{(1-x^2)^{5/2}} dx \quad \text{put } x = \sin y \quad \Rightarrow -5 \int \frac{\sin^4 y}{\cos^4 y} dy = -5 \int \tan^4 y \cdot dy$$

$$dx = \cos y \cdot dy$$

$$u = \tan y$$

$$du = \sec^2 y \cdot dy$$

$$\frac{du}{1+u^2} = dy$$

$$\Rightarrow -5 \int \frac{u^4}{1+u^2} du = -5 \int \left(u^2 - 1 + \frac{1}{1+u^2} \right) du$$

$$\begin{array}{r} u^2 - 1 \\ 1+u^2 \overline{) u^4} \\ \underline{u^4 + u^2} \\ 0 - u^2 - 1 \\ \underline{-u^2 - 1} \\ 0 + 1 \end{array}$$

$$= -5 \left(\frac{u^3}{3} - u + \int dy \right) = -5 \left(\frac{\tan^3 y}{3} - \tan y + y \right)$$

$$= -5 \left(\frac{\tan^3(\sin^{-1} x)}{3} - \tan(\sin^{-1} x) + \sin^{-1} x \right)$$

put $u = \tan y$
 $du = \sec^2 y \cdot dy$
 $\frac{du}{1+u^2} = dy$

★ $\rightarrow$ change of variable $\rightarrow \int f(x) \cdot dx$

put $x = g(u)$
 $dx = g'(u) du$

$$\Rightarrow \int f(g(u)) \cdot g'(u) \cdot du$$

& $f(g(u)) \cdot g'(u)$ might be easily integrable.

$\rightarrow$ Change of variable by trig substitution is extremely good.

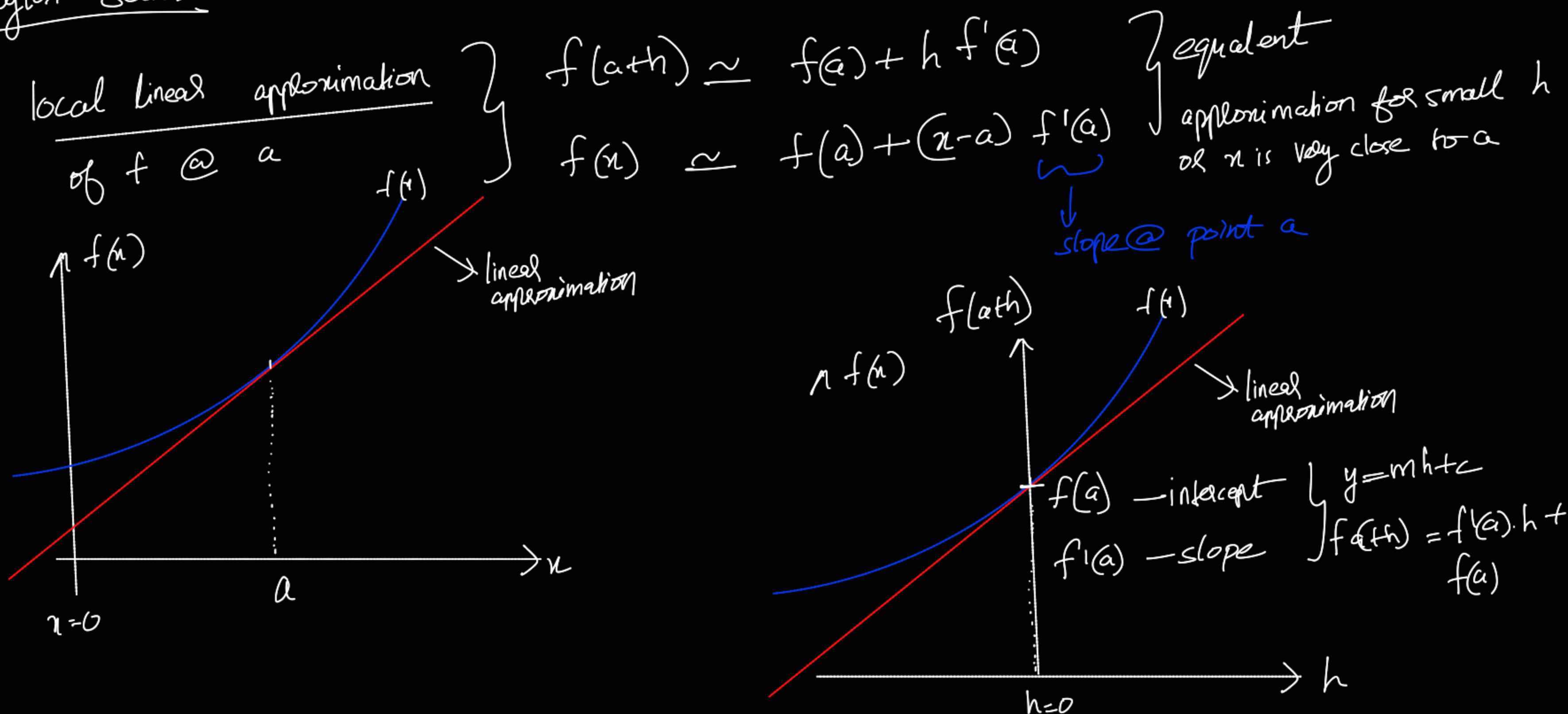
eg. $\int \frac{1}{1+x^2} dx$

$$= \int dy = y = \underline{\tan^{-1} x}$$

put $x = \tan y$
 $dx = \sec^2 y \cdot dy$
 $\frac{dx}{1+x^2} = dy$

★ $\int \frac{\text{Poly}(x)}{\text{Poly}(x)}$ if numerator has higher degree then polynomial division.

Taylor Series



→ how to improve → Add more polynomial terms.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

$\left. \begin{array}{l} \text{nth degree polynomial approximation} \\ \text{about point } a \end{array} \right\}$

In general

$$f(a) = c_0$$

$$f'(a) = c_1$$

$$f''(a) = 2c_2$$

$$f'''(a) = 3 \cdot 2 c_3$$

$$f^{(n)}(a) = n! c_n$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

→ We take n degree polynomial so the approximation will have n terms

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Taylor series approximation of $f(x)$ about point a

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

1. Write down the Taylor series for the following functions centered at a .

(a) $f(x) = e^{x^2}$ centered at $a = 0$.

(b) $f(x) = x^3$ centered at $a = 1$.

b) $f(1) = 1$

$f'(1) = 3x^2|_{x=1} = 3$

$f''(1) = 6x|_{x=1} = 6$

$f'''(1) = 6$

$$x^3 = 1 + (x-1)3 + (x-1)^2 \cdot 3 + (x-1)^3 = x^3$$

Polynomial Taylor series is the polynomial itself



(a) $f(0) = e^0 = 1$

$f'(0) = e^{x^2} \cdot 2x|_{x=0} = 0$

$f''(0) = 2x(e^{x^2} \cdot 2x) + e^{x^2} \cdot 2|_{x=0} = 2$

put $x = x^2$ in TS of e^x

$$= e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$$

Since Power series is unique this must be the Taylor series

Taylor Series of e^x

$$\left. \begin{array}{l} f'(x) = e^x \\ f^n(x) = e^x \end{array} \right\} \Rightarrow \begin{array}{l} f'(0) \\ f^n(0) = 1 \end{array} \quad \Rightarrow \quad \text{Taylor series} = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

★

Q Write to order (upto x^6) for $f(x) = e^{-4x^3}$

$$e^x = 1 + x + \frac{x^2}{2} + \dots \quad \Rightarrow \quad \begin{array}{l} e^{-4x^3} = 1 + (-4x^3) + \frac{(-4x^3)^2}{2!} \\ e^{-4x^3} = 1 - 4x^3 + \frac{16x^6}{2!} \end{array}$$

Q Taylor series expansion of $x^2 e^{-x}$ at $a=0$

★

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad \Rightarrow \quad x^2 e^{-x} = x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$$

→ For any questions involving e^x , try Taylor of e^x first

Constrained Optimization

Problem

$$\begin{aligned} &\text{minimize } f(\mathbf{x}) \\ &\text{subject to: } \mathbf{x} \in \mathbb{R}^n \end{aligned}$$

Unconstrained
revision

First Order Necessary Conditions

If $\mathbf{x}^*$ is a local minimizer of $f(\mathbf{x})$ and $f(\mathbf{x})$ is continuously differentiable in an open neighbourhood of $\mathbf{x}^*$, then

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

That is, $f(\mathbf{x})$ is **stationary** at $\mathbf{x}^*$

Second Order Necessary Conditions

If $\mathbf{x}^*$ is a local minimizer of $f(\mathbf{x})$ and $\nabla^2 f(\mathbf{x})$ is continuously differentiable in an open neighbourhood of $\mathbf{x}^*$, then

gradient $\nabla f(\mathbf{x}^*) = \mathbf{0}$

Hessian $\nabla^2 f(\mathbf{x}^*)$ is positive **semi definite** $H(\mathbf{x}) \succeq \mathbf{0}$

Necessary but
not sufficient

Second Order Sufficient Conditions

Suppose that $\nabla^2 f(\mathbf{x})$ is continuously differentiable in an open neighbourhood of $\mathbf{x}^*$. If the following two conditions are satisfied, then $\mathbf{x}^*$ is a local minimum of $f(\mathbf{x})$.

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

$$\nabla^2 f(\mathbf{x}^*) \text{ is positive definite}$$

Not necessary } For something to be
but sufficient } local min this is not
absolutely required. But if it is satisfied then
it is absolutely a local min



Direct Substitution

→ Limitation → if we cannot solve the equality constrain for a closed form solution of one of the variables eg: $x^4 + 5x^3y + y^2x + x^6 + 5 = 0$
→ Cannot separate out one of the variable

- When the constraint(s) are **equalities**, we can convert the problem from a constrained optimization to an unconstrained optimization problem by substituting for some of the variables.

eg. Min. $f(\mathbf{x}) = 4x_1^2 + 5x_2^2$
Subject to $2x_1 + 3x_2 = 6$
 $\Rightarrow x_1 = \frac{6-3x_2}{2}$

$$\begin{aligned} \Rightarrow f(x) &= (6-3x_2)^2 + 5x_2^2 = 36 - 36x_2 + 9x_2^2 + 5x_2^2 \\ \frac{\partial f(x)}{\partial x_2} &= 0 \Rightarrow 14x_2^2 - 36x_2 + 36 = 0 \end{aligned}$$

Solve for x_2 and then x_1 .

The Lagrangian Approach

$\mathbf{x} \rightarrow$ vector $[x_1, x_2, x_3, \dots, x_n]_{n \times 1} \rightarrow n$ variables

Max/Min

$$f(\mathbf{x}) = 0 \quad | \quad g(\mathbf{x}) = c, \text{ then use}$$

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \quad (1)$$

$n+1$ variables

$$\frac{\partial L(\mathbf{x}^*)}{\partial \mathbf{x}} = 0 \quad \frac{\partial L(\lambda^*)}{\partial \lambda} = 0 \quad (2)$$

→ We are not max/min the Lagrangian, we are just interested in the critical values of Lagrangian function
→ original constrain

$$\Rightarrow \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \lambda \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}$$

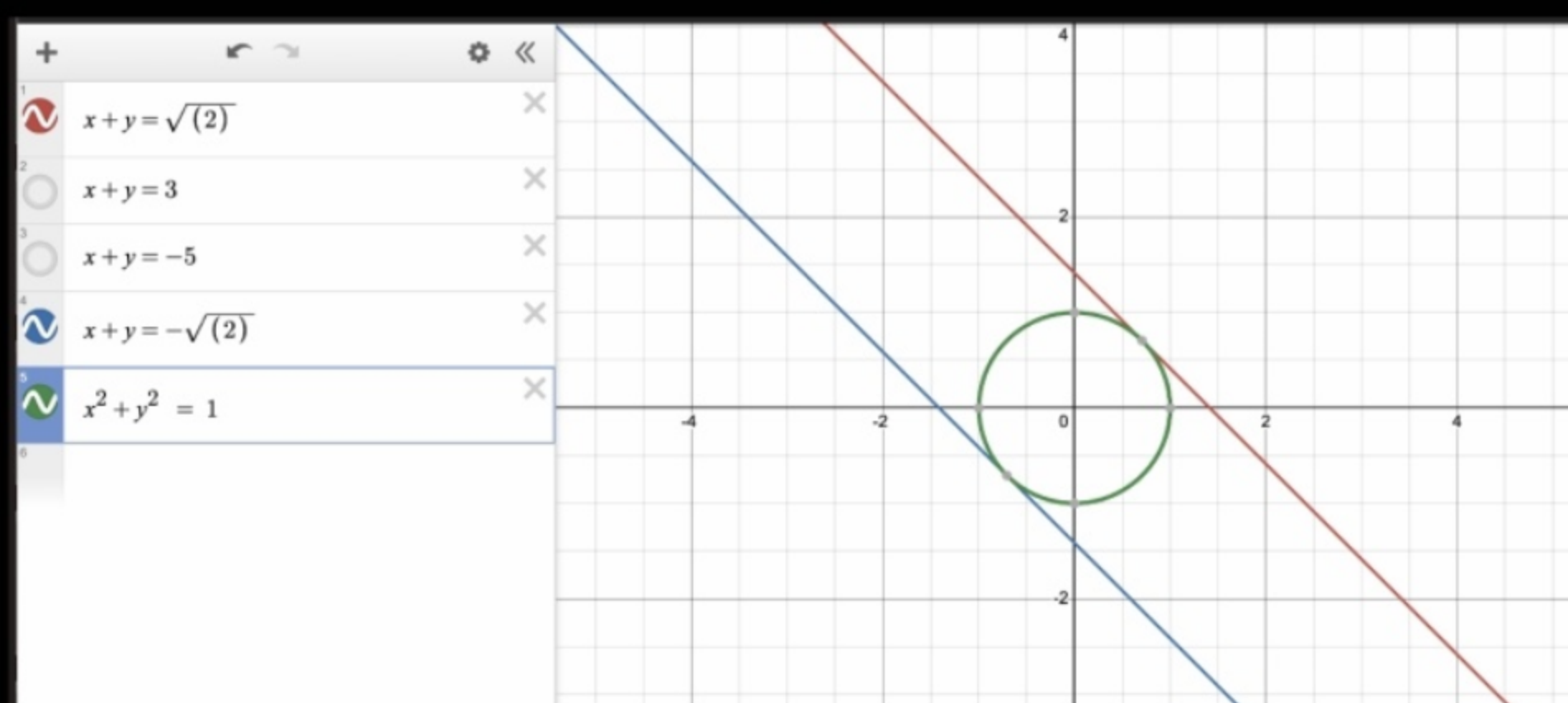
gradient of f is parallel to gradient of g
 $\nabla f \parallel \nabla g$

ⓑ Substitute & find min/max

$\phi \quad f(x,y) = x+y \quad \rightarrow \text{find min \& max } f(x,y) \quad \Rightarrow \frac{\partial L}{\partial x} = 0 \quad \Rightarrow 1 - 2dx = 0 \quad \Rightarrow dx = 1/2$
 $x^2 + y^2 = 1$
 $g(x,y) = x^2 + y^2 - 1 = 0$
 $L(x,y,d) = x+y - d(x^2 + y^2 - 1)$
 $\frac{\partial L}{\partial y} = 0 \quad \Rightarrow 1 - 2dy = 0 \quad \Rightarrow dy = 1/2$
 $\frac{\partial L}{\partial d} = 0 \quad \Rightarrow x^2 + y^2 - 1 = 0 \quad \left. \begin{array}{l} x^2 + y^2 = 1 \\ (\frac{1}{2d})^2 \cdot 2 = 1 \end{array} \right\}$

$\Rightarrow d = 1/\sqrt{2} \quad , \quad y = 1/\sqrt{2} \quad , \quad x = 1/\sqrt{2}$
 $d = -1/\sqrt{2} \quad , \quad y = -1/\sqrt{2} \quad , \quad x = -1/\sqrt{2}$

$\Rightarrow \text{min} = (x,y) = -1/\sqrt{2}, -1/\sqrt{2}$
 $\text{max} = (x,y) = 1/\sqrt{2}, 1/\sqrt{2}$



$\phi \quad f(x,y) = 4x^2 + 9y^2 \quad | \quad xy = 4$

$L(x,y,d) = 4x^2 + 9y^2 - d(xy - 4)$

$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow 8x - dy = 0 \quad \Rightarrow dx = 8y$
 $\frac{\partial L}{\partial y} = 0 \quad \Rightarrow 18y - dx = 0 \quad \Rightarrow \frac{x}{y} = \frac{18y}{8x}$
 $\frac{\partial L}{\partial d} = 0 \quad \Rightarrow xy = 4 \quad \Rightarrow \frac{xy}{8x^2} = \frac{4}{18y^2}$

$8x^2 = 18 \left(\frac{4}{x}\right)^2$

$8x^4 = 18 \cdot 16$

$x^4 = 36$

$x^2 = \pm 6$

$x = \pm\sqrt{6}, \pm i\sqrt{6}$

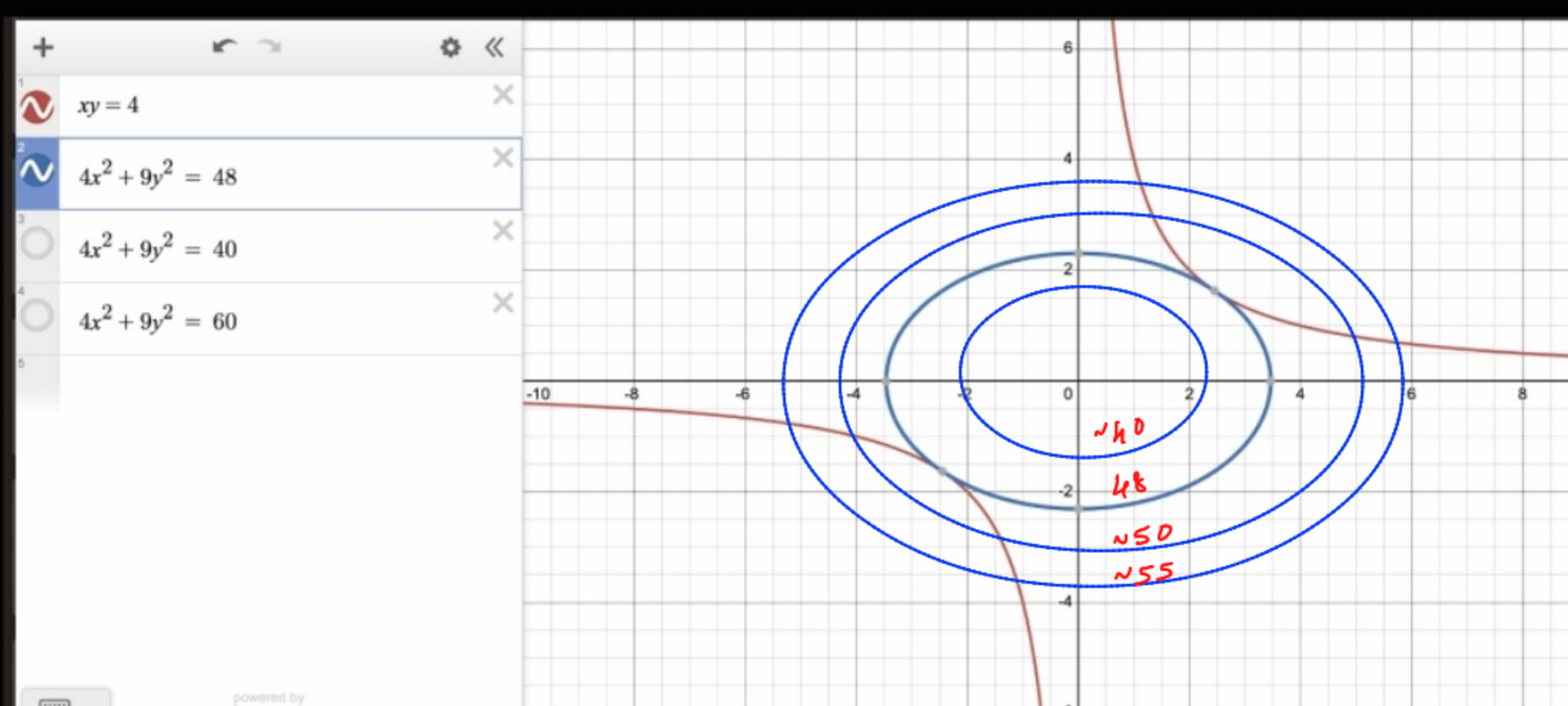
$y = \pm \frac{4}{\sqrt{6}}, \pm \frac{i4}{\sqrt{6}}$

not possible since $x,y \in \mathbb{R}$

$f(\sqrt{6}, \frac{4}{\sqrt{6}}) = 4 \cdot 6 + 9 \cdot \frac{16}{6} = 24 + 24 = 48$

$f(-\sqrt{6}, -\frac{4}{\sqrt{6}}) = 48$

This is minima because $f(x,y)$ is unbound given constrain



crossing point is never optimum
touching point is optimum

Geometric Intuition for Lagrangian method

objective $\rightarrow \text{Min/Max } f(\bar{x}) \quad | \quad g(\bar{x}) = 0$

$\bar{x}$ is a vector in $\mathbb{R}^n$

$\Rightarrow g(x) = 0 \quad \& \quad f(x) = k$ are surfaces in $\mathbb{R}^n$

$\rightarrow$ These surfaces may cross each other or just touch

$\rightarrow$ Assume we are drawing level curves of $f(x) = k$ using different values of k

$\Rightarrow$ Then min/max will be touching point & not crossing point

$\Rightarrow$ gradient of f @ x^* = gradient of g @ x^* } touching point x^*

$\nabla f = \nabla g$

$\Rightarrow L(x,y,d) = f(x) - d g(x)$

d) extreme values of $f(x,y) = xe^y$ | $x^2+y^2=2$

$$L(x,y,d) = xe^y - d(x^2+y^2-2)$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow e^y - 2dx = 0$$

$$e^y = 2dx$$

$$\Rightarrow y+y^2=2$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow xe^y - 2dy = 0$$

$$e^y = 2dy/x$$

$$y^2 - 2 + y = 0$$

$$(y+2)(y-1) = 0$$

$$\frac{\partial L}{\partial d} = 0 \Rightarrow x^2+y^2=2$$

$$\Rightarrow x = \frac{y}{x}$$

$$x^2 = y$$

$$y = -2, 1$$

$$x = \pm\sqrt{2} \pm 1$$

$$f(1,1) = e \text{ Max}$$

$$f(-1,1) = -e \text{ Min}$$

e) extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2+y^2=1$

$$L(x,y,d) = x^2 + 2y^2 - d(x^2+y^2-1) = 0$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2x - 2dx = 0 \Rightarrow \frac{\partial L}{\partial y} = \frac{x}{y}$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow 4y - 2dy = 0 \Rightarrow x=0 \text{ or } y=0$$

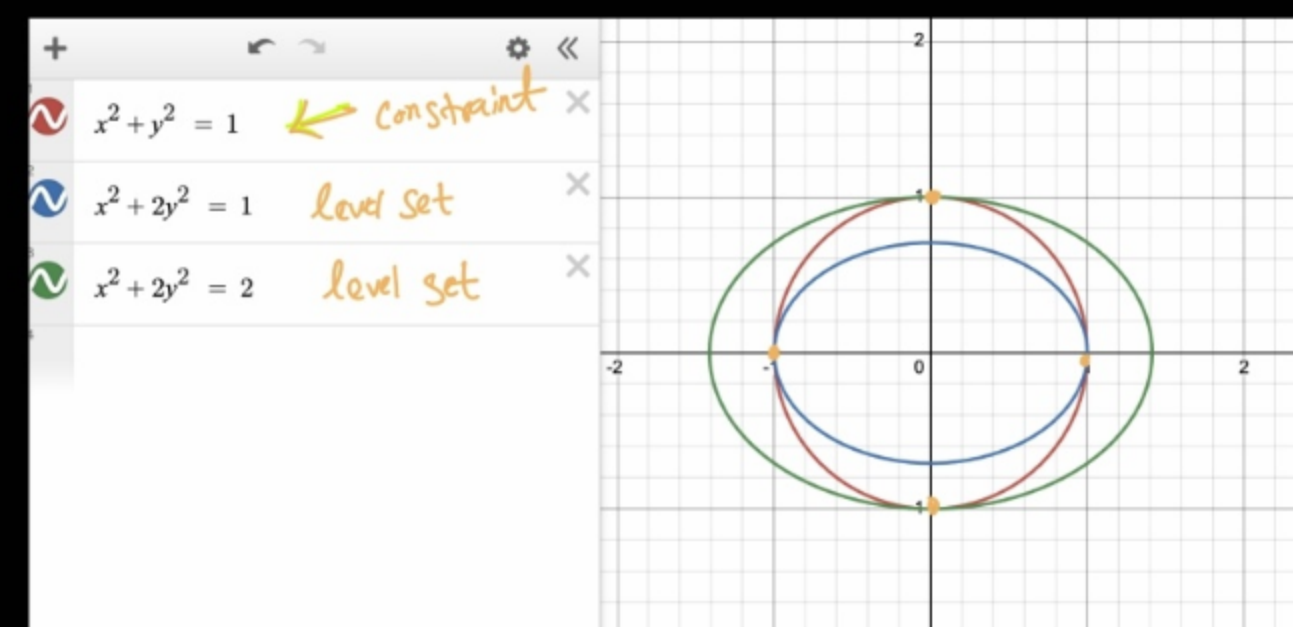
$$\frac{\partial L}{\partial d} = 0 \Rightarrow x^2+y^2=1$$

$$x,y = (0,1), (0,-1), (1,0), (-1,0)$$

$$f(0,1) = 2 \quad f(1,0) = 1$$

$$f(0,-1) = 2 \quad f(-1,0) = 1$$

$$\begin{aligned} 2x(1-d) &= 0 & d=1 \text{ or } x=0 \\ 2y(2-d) &= 0 & d=2 \text{ or } y=0 \end{aligned}$$



max $x^T A x$
s.t. $x^T x = 1$

A is symmetric

$$L(x,d) = x^T A x - d[x^T x - 1]$$

$$\nabla_x L(x,d) = x^T (A + A^T) - 2dx^T$$

$$\frac{\partial L}{\partial d} = x^T x - 1 = 0$$

$$A + A^T = 2A \Rightarrow A \text{ is sym given}$$

$$\Rightarrow 2x^T A - 2dx^T = 0 \Rightarrow x^T A = dx^T$$

$$2x^T (A - dI) = 0 \Rightarrow Ax = dx$$

$$x^T \neq 0 \Rightarrow d \text{ is eigen value}$$

$$x \rightarrow \text{eigen vector}$$

Side quest

$$x^T A x \rightarrow \text{scalar}$$

$$(x^T A) x$$

$$\underbrace{x^T A}_{\text{row}} + \underbrace{A^T x}_{\text{column } x \text{ converted to row}}$$

$$\Rightarrow x^T (A + A^T) \text{ row}$$

$$= (A + A^T) x \text{ column}$$

Numerical Layout FIVIR
denominator convert

A is symmetric $x^T (Ax) = x^T d x = d x^T x = d$

$x^T A x$ quadratic form
→ maximum value is the maximum eigen value @ the unit eigen vector corresponding to this maximum eigen value.

→ Minimum value is the minimum eigen value @ the unit eigen vector corresponding to this minimum eigen value.



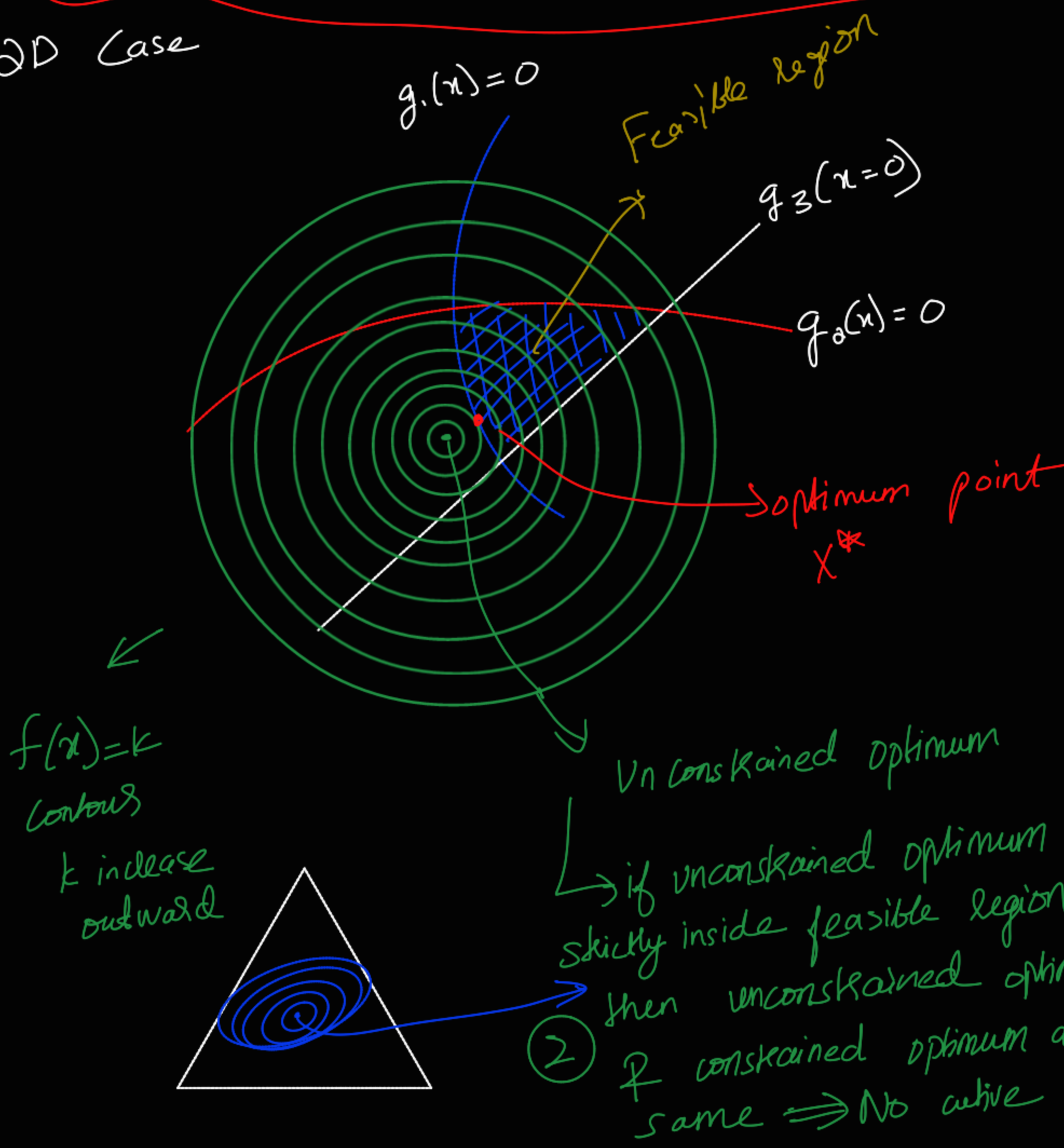
Here if instead of $x^T x = 1$ they give $x^T x = k$ then also eigen value max = max of $x^T A x$
min = min of $x^T A x$
but $x = \sqrt{k} \hat{x}$ $\hat{x}$ unit eigen vector of the corresponding eigen value.

INEQUALITY CONSTRAINED OPTIMIZATION

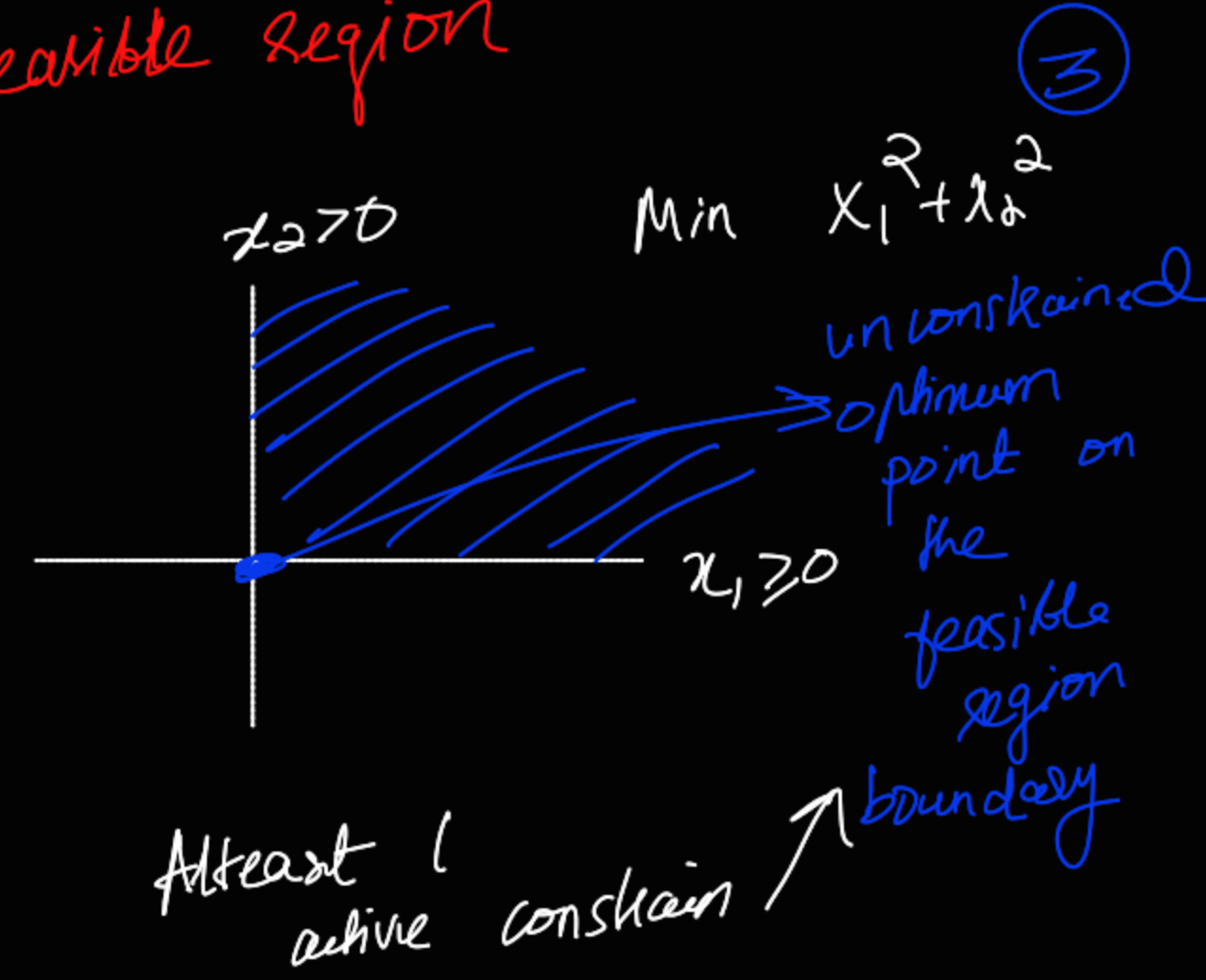
Minimise $f(x)$ | $g_i(x) \leq 0 \quad i=1,2,\dots,m$

if we need to maximise, then we can take $-f(x)$ & minimise it

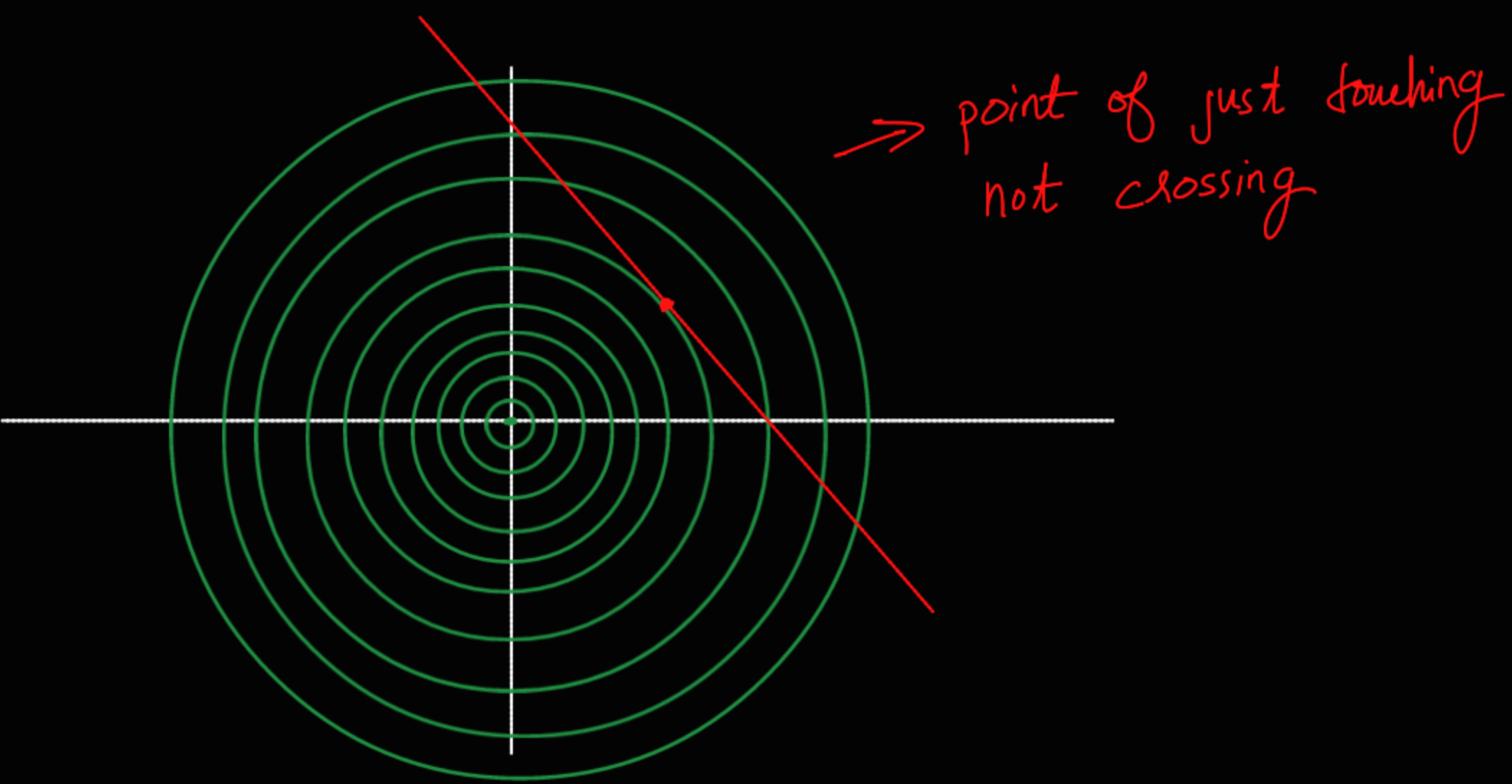
2D Case



- All constraints are satisfied
- $g_i(x^*)=0$ } Active constraint
- other constraint may or may not be active
- At least one is active if ① unconstrained solution is outside feasible region



eg. Min $f(x) = 4x_1^2 + 4x_2^2$



The Karush-Kuhn-Tucker Theorem

KKT theorem

$$\text{Minimize } f(x) \quad | \quad \underbrace{g(x) \leq 0}_{\text{inequality}} ; \underbrace{h(x) = 0}_{\text{equality}} \\ \underbrace{\hspace{10em}}_{\text{constraints}}$$

if x^* is a local minima, the the following **necessary** conditions hold

$$L(x, p, d) = f(x) + p g(x) + d h(x)$$

$$\nabla_x L(x, p, d) = 0 \Rightarrow \nabla f(x^*) + p \nabla g(x^*) + d \nabla h(x^*) = 0 \rightarrow \text{stationarity (1)}$$

$$g(x^*) \leq 0 \quad \left. \begin{array}{l} \text{feasibility (2)} \end{array} \right\} \begin{array}{l} \text{inequality} \\ \text{equality} \end{array}$$

$$h(x^*) = 0$$

$$p \geq 0 \rightarrow \text{Non negativity (4)}$$

$$p g(x^*) = 0 \rightarrow \text{Complementary slackness (5)}$$

KKT conditions

Intuition $\rightarrow$ if a particular inequality is active $g(x^*)$ only then can p be non zero. & vice versa

★ optimal solution $\Rightarrow$ KKT conditions } necessary condition only
KKT conditions $\nRightarrow$ optimal solution

Note) Hessian test fails here to determine whether minima if KKT is satisfied — since the function may decrease in some direction but not feasible to optimise x^* in that direction due to constrain. Same is true with previous method of Lagrangian optimisation for equality constrain since. The interesting direction is the direction of feasibility, still we cannot say Hessian is +ve definite in that direction, since constrained optima $\neq$ unconstrained optima.

$$\min_x f(x)$$

s. t.

$$h(x) = 0$$

$$g(x) \leq 0$$

$$\mathcal{L} = f(x) + \lambda h(x) + \mu g(x)$$

KKT conditions are:

$$\nabla_x f(x) + \lambda \nabla_x h(x) + \mu \nabla_x g(x) = 0$$

$$h(x) = 0$$

$$g(x) \leq 0$$

$$\mu g(x) = 0$$

$$\mu \geq 0$$

$$(1a) \rightarrow \text{stationary cond}^n$$

$$(1b) \rightarrow \text{feasibility cond}^n$$

$$(1c) \rightarrow \text{feasibility cond}^n$$

$$(1d) \rightarrow \text{Complementary slackness cond}^n$$

$$(1e) \rightarrow \text{Non negativity cond}^n$$

The following optimization problem is given:

$$\begin{aligned} &\text{maximize } x_1^2 + 4x_2^2 \\ &\text{subject to } x_1^2 + 2x_2^2 \leq 2 \end{aligned}$$

Derive the KKT conditions and find the set of points satisfying these conditions

$$\begin{aligned} \Rightarrow 2\mu x_1 = 2x_1 &\Rightarrow 2x_1(\mu - 1) = 0 \Rightarrow \mu = 1 \text{ or } x_1 = 0 \\ 4\mu x_2 = 8x_2 &\Rightarrow 4x_2(\mu - 2) = 0 \Rightarrow \mu = 2 \text{ or } x_2 = 0 \\ \mu = 1, x_2 = 0, &\Rightarrow \text{inequality active} \Rightarrow x_1^2 + 2x_2^2 - 2 = 0 \Rightarrow x_1 = \pm\sqrt{2} \end{aligned}$$

$$\begin{aligned} a) \mu = 1, x_2 = 0, x_1 = \pm\sqrt{2} & \quad f(\pm\sqrt{2}, 0) = 2 \\ b) \mu = 2, x_1 = 0, x_2 = \pm 1 & \quad f(0, \pm 1) = 4 \text{ (maxima)} \\ c) \mu = 0, x_1 = 0, x_2 = 0 & \quad f(0, 0) = 0 \text{ (minima)} \end{aligned}$$

$$f(x_1, x_2) = -x_1^2 - 4x_2^2 \quad | \quad x_1^2 + 2x_2^2 \leq 2$$

$$L(x_1, x_2, \mu) = -x_1^2 - 4x_2^2 + \mu(x_1^2 + 2x_2^2 - 2)$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \begin{cases} -2x_1 + 2\mu x_1 = 0 \\ -8x_2 + 4\mu x_2 = 0 \end{cases} \quad (1)$$

$$\mu \geq 0 \quad (5)$$

$$x_1^2 + 2x_2^2 - 2 \leq 0 \quad (2, 3)$$

$$\mu(x_1^2 + 2x_2^2 - 2) = 0 \quad (4)$$

$$\phi \text{ Min } f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 \quad | \quad \begin{aligned} g_1(x) &= x_1 - x_2 - 2x_3 - 12 \leq 0 \\ g_2(x) &= x_1 + 2x_2 - 3x_3 - 8 \leq 0 \end{aligned}$$

$$L(x_1, \mu_1, \mu_2) = x_1^2 + 2x_2^2 + 3x_3^2 + \mu_1(x_1 - x_2 - 2x_3 - 12) + \mu_2(x_1 + 2x_2 - 3x_3 - 8)$$

$$\nabla_x L(x_1, \mu_1, \mu_2) = 0 \Rightarrow \begin{cases} 2x_1 + \mu_1 + \mu_2 = 0 \\ 4x_2 - \mu_1 + 2\mu_2 = 0 \\ 6x_3 - 2\mu_1 - 3\mu_2 = 0 \end{cases} \quad (1) \text{ stationary cond}^n$$

$$\mu_1 \geq 0; \mu_2 \geq 0 \rightarrow (2) \text{ Non negativity cond}^n$$

$$\begin{aligned} \mu_1(x_1 - x_2 - 2x_3 - 12) &= 0 \\ \mu_2(x_1 + 2x_2 - 3x_3 - 8) &= 0 \end{aligned} \quad (3) \text{ Complementarity slackness}$$

$$\begin{aligned} x_1 - x_2 - 2x_3 - 12 &\leq 0 \\ x_1 + 2x_2 - 3x_3 - 8 &\leq 0 \end{aligned} \quad (4) \text{ feasibility cond}^n$$

$$a) \mu_1 = 0$$

$$\begin{aligned} \Rightarrow 2x_1 + \mu_2 &= 0 \\ 4x_2 + 2\mu_2 &= 0 \\ 6x_3 - 3\mu_2 &= 0 \end{aligned} \quad \left| \begin{aligned} \mu_2 &= -2x_1 \\ \mu_2 &= -2x_2 \\ \mu_2 &= 2x_3 \end{aligned} \right. \Rightarrow x_1 = x_2 = -x_3$$

$$\mu_2(x_1 + 2x_2 - 3x_3 - 8) = 0$$

$$\Rightarrow \mu_2 = 0 \quad \left| \begin{aligned} x_1 &= x_2 = 4/3 \\ x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \right. \quad \text{or} \quad \begin{aligned} \mu_2 &= -8/3 \\ x_3 &= -4/3 \end{aligned}$$

$$b) \mu_2 = 0 \Rightarrow \begin{aligned} 2x_1 + \mu_1 &= 0 \\ 4x_2 - \mu_1 &= 0 \\ 6x_3 - 2\mu_1 &= 0 \end{aligned} \Rightarrow \begin{aligned} \mu_1 &= -2x_1 \\ \mu_1 &= 4x_2 \\ \mu_1 &= 3x_3 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -2x_2 \\ x_3 &= \frac{4}{3}x_2 \end{aligned}$$

$$\mu_1(-2x_2 - x_2 - \frac{8}{3}x_2 - 12) = 0$$

$$\Rightarrow \mu_1 = 0 \quad \left| \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \right. \quad \text{or} \quad \begin{aligned} -\frac{17}{3}x_2 &= 12 \\ x_2 &= -\frac{36}{17}, \quad x_1 = +\frac{72}{17} \\ x_3 &= -\frac{48}{17} \end{aligned}$$

c) $\mu_1 \neq 0, \mu_2 \neq 0$

$\Rightarrow$ System of 5 linear equations.

$$\begin{cases} 2x_1 + \mu_1 + \mu_2 = 0 \\ 4x_2 - \mu_1 + 2\mu_2 = 0 \\ 6x_3 - 2\mu_1 - 3\mu_2 = 0 \\ x_1 - x_2 - 2x_3 - 12 = 0 \\ x_1 + 2x_2 - 3x_3 - 8 = 0 \end{cases} \begin{matrix} 5 \text{ eq} \\ 5 \text{ unknown} \end{matrix}$$

★ Convex optimisation programming $\rightarrow$ Objective function & inequality & equality constraints are all convex functions.

Min $f(x) \mid g_i(x) \leq 0; h_j(x) = 0$ where $f(x), g_i(x), h_j(x)$ are all convex functions of vector x & i, j

then it is called a convex optimisation problem.

then KKT conditions are necessary & sufficient for x^* to be the global optima.

Duality $\rightarrow$ We need Min $f(x) \mid g(x) \leq 0$

define Lagrangian $L(x, \mu) = f(x) + \mu g(x)$

define $p(x) = \max_{\mu, \mu \geq 0} L(x, \mu) = \begin{cases} f(x), & x \rightarrow \text{satisfies constraint } g(x) \leq 0 \\ \infty, & x \rightarrow \text{does not satisfy constraint } g(x) \leq 0; g(x) > 0 \end{cases}$

here μ becomes 0 to maximise $L(x, \mu)$. If permitted μ becomes $-\infty$

here μ becomes ∞ to max $L(x, \mu)$

then $\min_x p(x) = \min_x \max_{\mu, \mu \geq 0} L(x, \mu) = \min_x \begin{cases} f(x), & x \rightarrow \text{satisfies constraint } g(x) \leq 0 \mu = 0 \\ \infty, & x \rightarrow \text{does not satisfy constraint } g(x) > 0 \mu \rightarrow \infty \end{cases}$

$\min_x \max_{\mu, \mu \geq 0} L(x, \mu) = \min_{x, g(x) \leq 0} f(x)$

Duality

x^* will avoid the infinite case to minimise so will satisfy the constraint

$\rightarrow$ The LHS is called a saddle point optimisation & hence the Min & Max can be interchanged with condition $g(x) \leq 0; \mu \geq 0$ Also called positivity condition

$\rightarrow$ It can be shown even for vector x that, the KKT conditions are a direct result of the duality of $\min_x f(x) \mid g(x) \leq 0$

$$\begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & g_i(x) \leq 0 \quad i = 1, \dots, m \\ & h_j(x) = 0 \quad j = 1, \dots, p \end{array}$$

Primal Problem

$$\begin{array}{ll} \max_{\lambda, \mu} & F(\lambda, \mu) \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

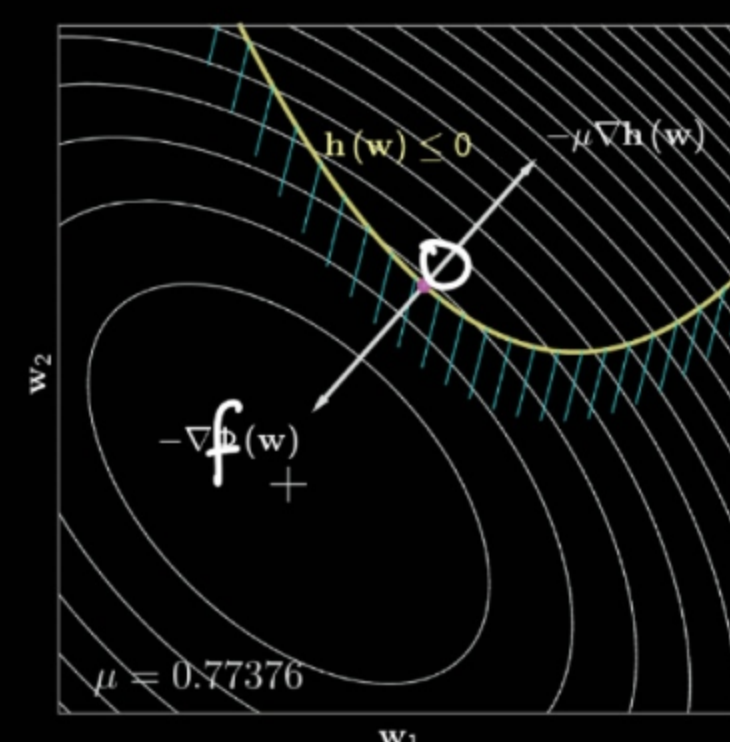
Dual Problem

Some intuitions on the KKT conditions

$$\begin{array}{ll} \min_w & \Phi(x) \\ \text{s.t.} & h(w) \leq 0 \end{array}$$

Ball rolling down a valley blocked by a fence

- $-\nabla \Phi$ is the gravity
- $-\mu \nabla h$ is the force of the fence. Sign $\mu \geq 0$ means the fence can only "push" the ball.



Geometric Intuition

$$\begin{array}{l} \text{Min } f(x) \\ \text{s.t. } g_1(x) \leq 0 \\ \quad g_2(x) \leq 0 \\ \quad g_3(x) \leq 0 \end{array}$$

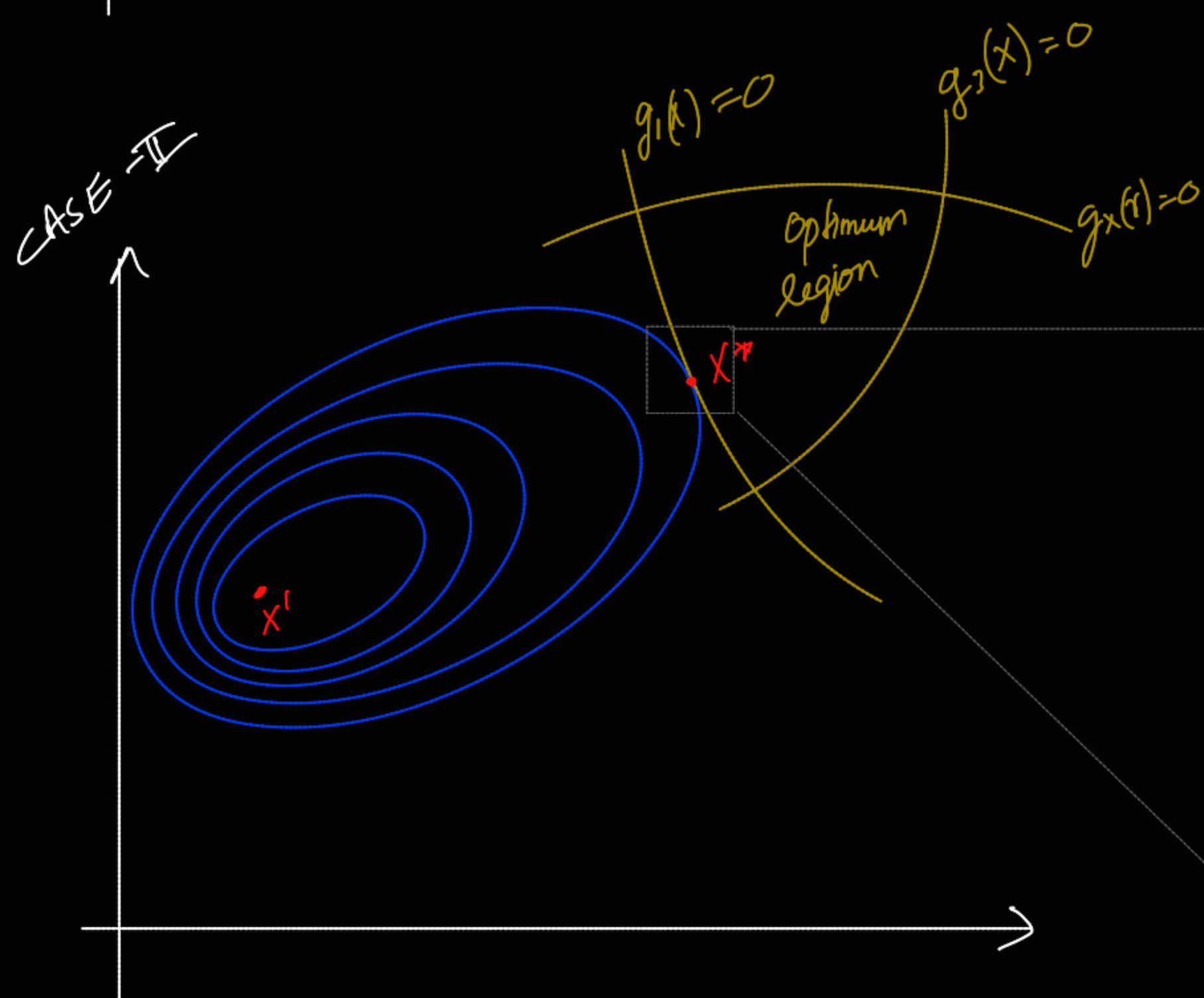
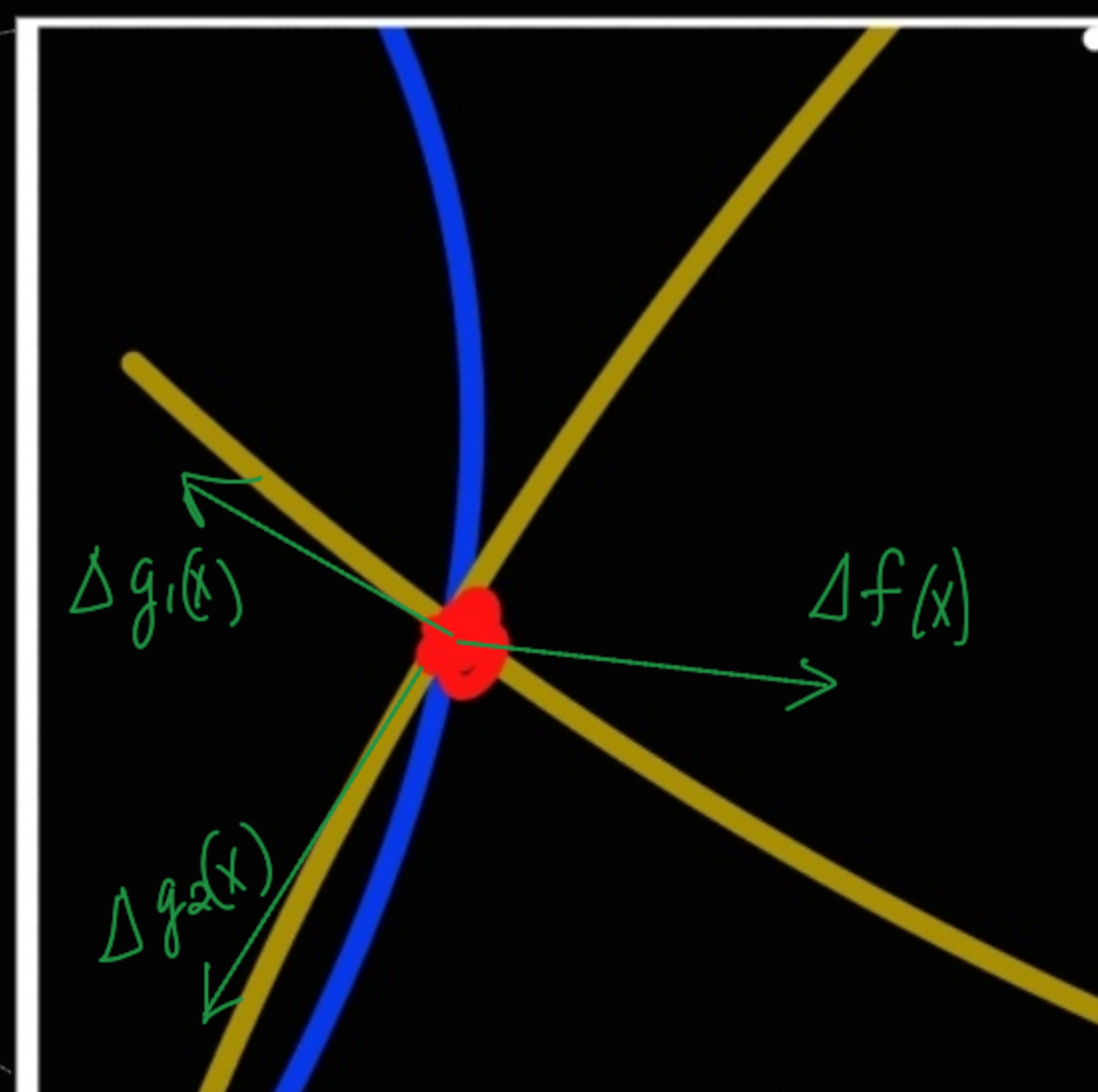
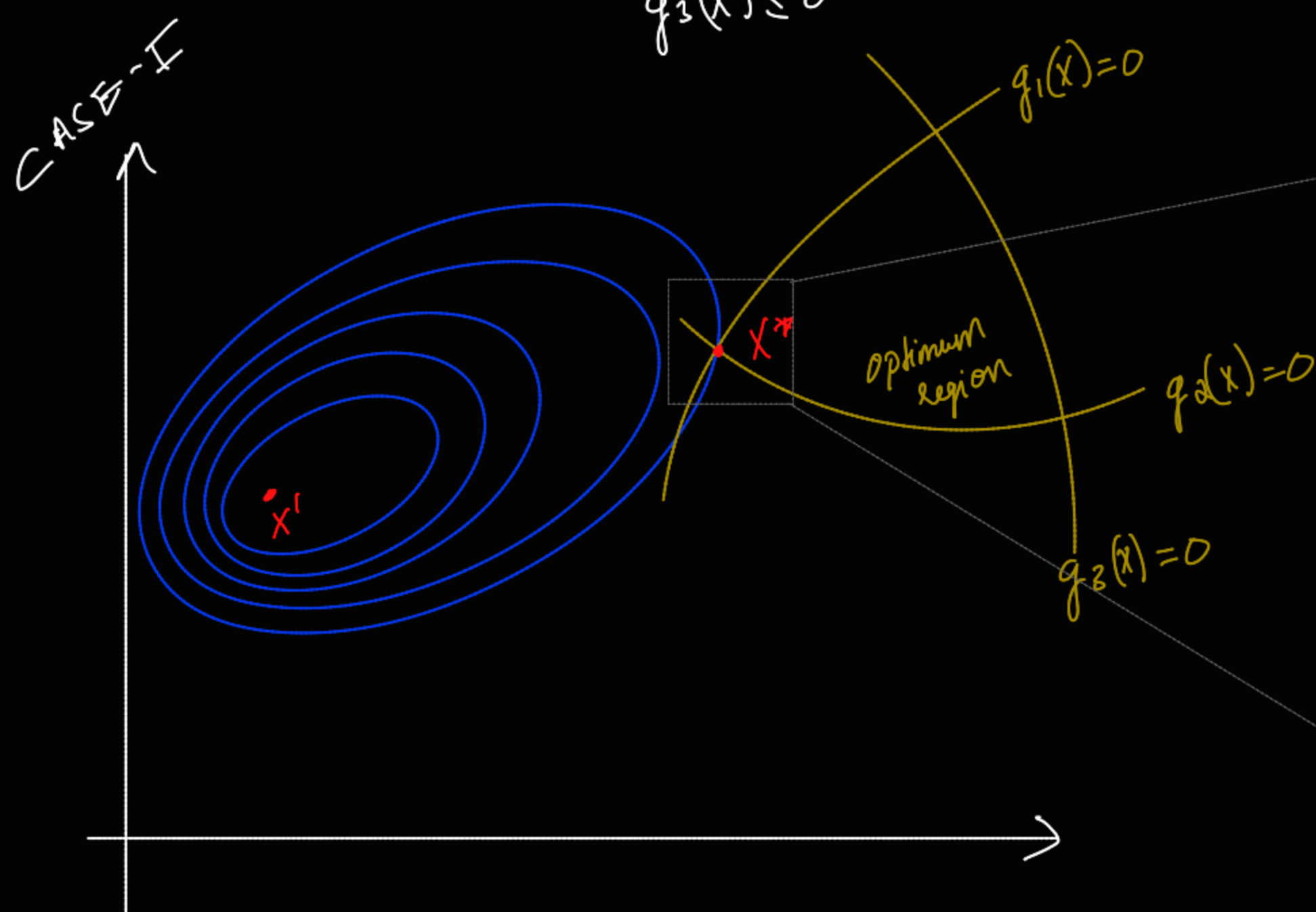
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

unconstrained
optima = x^*

constrained
optima = x^*

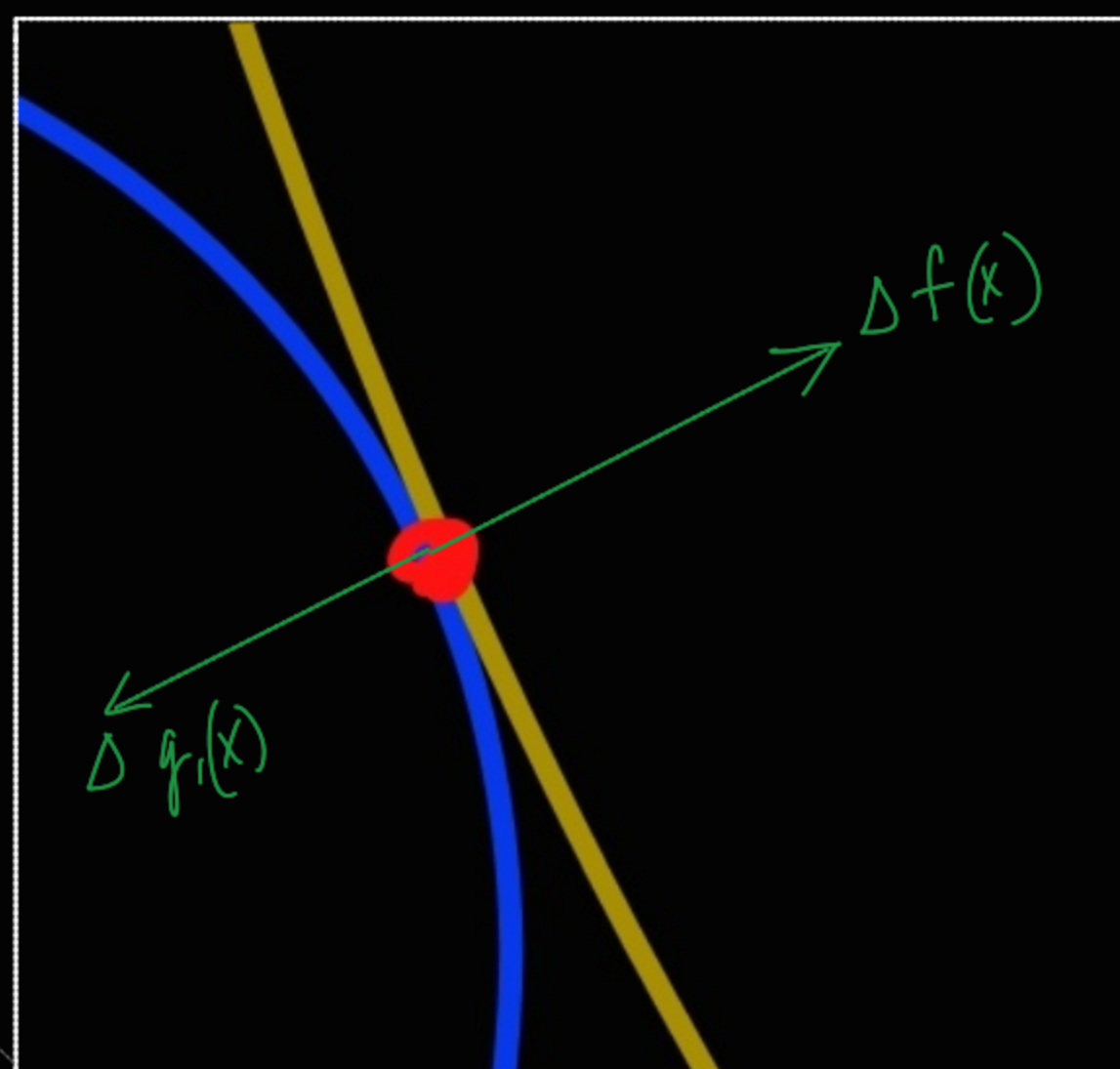
$$\text{from } \Delta_x L(x, \mu) = 0 \Rightarrow \Delta f(x) = -\mu_1 \Delta g_1(x) - \mu_2 \Delta g_2(x)$$

$\mu_3 = 0$
Since
inactive
constraint.



Intuition $\rightarrow \Delta f(x)$ in the general oppo.

- direction of $\Delta g_i(x) \rightarrow$ where $g_i(x)$ is active
- otherwise $\mu_j = 0$ for $g_j(x)$ inactive



Maxima, Minima & Saddle point

For Single valued function $f(x): \mathbb{R} \rightarrow \mathbb{R}$

- ① Stationary $\Rightarrow f'(x) = 0$
 - $\rightarrow$ ① Maxima $f''(x) < 0$
 - $\rightarrow$ ② Minima $f''(x) < 0$
 - $\rightarrow$ ③ Sta. Inflexion point $f''(x) = 0$
 - ② Inflexion point $f''(x) = 0$; $f'(x) \neq 0$
 changes sign
 - ③ Critical point (x_0) $\begin{cases} f'(x_0) = 0 \text{ or} \\ f'(x_0) \text{ doesn't exist } \pm \infty \end{cases}$
- all stationary points are critical points

For convex function $f(x)$

- Hessian is PSD $\rightarrow$ necessary & sufficient
- Hessian is PD $\rightarrow$ sufficient not necessary

For concave function $f(x)$

- Hessian is NSD $\rightarrow$ necessary & sufficient
- Hessian is ND $\rightarrow$ sufficient not necessary

Problem

minimize $f(x)$
subject to: $x \in \mathbb{R}^n$

First Order Necessary Conditions

If x^* is a local minimizer of $f(x)$ and $f(x)$ is continuously differentiable in an open neighbourhood of x^* , then

$$\nabla f(x^*) = 0$$

That is, $f(x)$ is stationary at x^*

Second Order Necessary Conditions

If x^* is a local minimizer of $f(x)$ and $\nabla^2 f(x)$ is continuously differentiable in an open neighbourhood of x^* , then

$$\nabla f(x^*) = 0$$

$$\text{Hessian } \nabla^2 f(x^*) \text{ is positive semi definite } H(x) \geq 0$$

Second Order Sufficient Conditions

Suppose that $\nabla^2 f(x)$ is continuously differentiable in an open neighbourhood of x^* . If the following two conditions are satisfied, then x^* is a local minimum of $f(x)$.

$$\nabla f(x^*) = 0$$

$\nabla^2 f(x^*)$ is positive definite

Necessary but not sufficient

Not necessary } For some thing to be local min this is not absolutely required. But if it is satisfied then it is absolutely a local min

given determinant & trace of a Hessian matrix + $\nabla f(x) = 0$

$$\text{Let } |\nabla^2 f(x)| = D \quad \text{Trace}(\nabla^2 f(x)) = T \quad f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$$

- Case ① $D < 0$ $\rightarrow$ Atleast one -ve eigen value $\Rightarrow$ Not Positive Definite
- $\rightarrow$ if $n = \text{even}$ $\rightarrow$ Atleast one positive & negative eigen value $\rightarrow$ Not PD + Not ND
 - $\rightarrow$ if $T > 0$ $\rightarrow$ Atleast one positive & negative eigen value $\rightarrow$ Not PD + Not ND
- Case ② $D > 0$ $\rightarrow$ if $n = \text{odd}$ & $T < 0$ $\rightarrow$ Both +ve & negative eigen value $\Rightarrow$ Not PD + Not ND
- saddle point*

multivariate $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ $\bar{x} = [x_1, x_2, x_3, \dots, x_n]^T$

$$\nabla f(\bar{x}) = \begin{bmatrix} \frac{\partial f(\bar{x})}{\partial x_1} & \frac{\partial f(\bar{x})}{\partial x_2} & \dots & \frac{\partial f(\bar{x})}{\partial x_n} \end{bmatrix}$$

$$\nabla f(\bar{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Jacobian = gradient of Vector valued function $f(x) \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^n$
Hessian = Jacobian of gradient

$$\text{Hessian } \nabla^2 f(\bar{x}) = \begin{bmatrix} \frac{\partial^2 f(\bar{x})}{\partial x_1^2} & \frac{\partial^2 f(\bar{x})}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f(\bar{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\bar{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\bar{x})}{\partial x_2^2} & \dots & \frac{\partial^2 f(\bar{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\bar{x})}{\partial x_n \partial x_1} & \dots & \dots & \frac{\partial^2 f(\bar{x})}{\partial x_n^2} \end{bmatrix}$$

$\rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$

Sufficient condition not necessary.

- Maxima $\leftarrow \begin{cases} \text{gradient} = 0 \text{ vector } \& \\ \text{Hessian ND} \Leftrightarrow \text{all eigen values} < 0 \end{cases}$
- Minima $\leftarrow \begin{cases} \text{gradient} = 0 \text{ vector } \& \\ \text{Hessian PD} \Leftrightarrow \text{all eigen value} > 0 \end{cases}$
- Saddle point $\leftarrow \begin{cases} \text{both + \& -ve eigen values} \\ 0 \text{ eigen value} \rightarrow \text{test fails} \end{cases}$

for 2x2 matrix

$$\begin{aligned} |\nabla^2 f(x)| > 0 \& \nabla f(x) = 0 &\rightarrow \text{Maxima} \\ |\nabla^2 f(x)| > 0 \&\nabla f(x) = 0 &\rightarrow \text{Minima} \\ |\nabla^2 f(x)| < 0 \&\nabla f(x) = 0 &\rightarrow \text{saddle} \end{aligned}$$

$$\phi) \lim_{x \rightarrow 0} \left(\frac{e^{\sin x} - 1}{x} \right)^{\frac{1}{\tan x}} = e^k$$

$$k = \lim_{x \rightarrow 0} \frac{1}{\tan x} \cdot \ln \left(\frac{e^{\sin x} - 1}{x} \right) = \frac{1}{\tan x} \ln \left(\frac{1 + \sin x + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + \dots - 1}{x} \right)$$

$$= \frac{1}{\tan x} \ln \left(\frac{\sin x}{x} + \frac{\sin^2 x}{2x} + \dots \right) = \frac{1}{\tan x} \ln \left(\frac{\sin x}{x} \left(1 + \frac{\sin x}{2} + \dots \right) \right)$$

$$= \frac{1}{\tan x} \ln \left(\left(1 - \frac{x^2}{3!} + \frac{x^5}{5!} \right) \left(1 + \frac{1}{2} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \right) \right) \right)$$

$$= \frac{1}{\tan x} \ln \left(1 + \frac{x}{2} - \frac{x^2}{3!} + \dots \right) = \frac{x/2}{\tan x} = \frac{x/2}{x \left(1 - \frac{x^2}{3} + \dots \right)}$$

$$= 1/2 \quad \therefore k = \underline{\underline{1/2}}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$$

$$\ln \left(1 + \frac{x}{2} \right) = \frac{x}{2} + \frac{\left(\frac{x}{2} \right)^2}{2} + \dots$$

$$\tan x = x - \frac{x^3}{3} + \frac{x^5}{5}$$

$$= x \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \dots \right)$$